A-posteriori error estimation and adaptivity for elastoplasticity using the reciprocal theorem

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Abstract

We present a-posteriori error estimators and adaptive methods for the finite element approximation of nonlinear problems and especially elastoplasticity. The main characteristic of the proposed method is the introduction of duality techniques or in other notions the reciprocal theorem. For error estimation at an equilibrium point the nonlinear boundary value problem and an additional linearized dual problem are considered. The loading of the dual problem is specifically designed for capturing the influence of the errors of the entire domain to the considered variable. Our approach leads to easy computable refinement indicators for locally or integrally defined variables. For instationary problems as elastoplasticity, in a first step, we neglect the errors due to time discretization, and evaluate the error indicators within each time step for a stationary problem. The versatility of the presented framework is demonstrated with numerical examples.

1 Introduction

The efficient computation of nonlinear structural problems requires often frequent adaptation of the discretization during the whole computation. Therefore reliable and easy computable error indicators and remeshing procedures are essential especially for nonlinear problems. Depending on the considered example, the variables of interest and the related mesh refinement criteria are local or global variables, as local displacements, stresses or boundary tractions. The classical methods, however, enable the mesh adaptation only with respect to the global energy norm in the linear case or to the corresponding inner products in the nonlinear case. A recent review of the classical methods can be for example found in Verfürth [21] or Ainsworth and Oden [1]. The most common methods consider local Dirichlet or Neumann problems, interpolation estimates or postprocessing techniques for error estimation. The sum of the locally computed energy contributions gives in general a sufficient approximation to the global energy norm due to the Galerkin orthogonality or in other terms the best approximation property of the finite element method. As is meanwhile well known, for errors in other norms as energy norm the above concept is not sufficient [11]. In order to estimate the errors in local quantities or in other norms as the energy an additional dual problem must be introduced. Duality techniques or in other notions the reciprocal theorem has been already applied for linear problems in early seventies to a-posteriori estimates by Tottenham [20] and for a-priori estimates by Nitsche and Schatz [13]. Using the same techniques the authors introduced in [6] an error estimator and an adaptive algorithm for linear problems. Our approach gives easy computable refinement indicators of the Zienkiewicz and Zhu [24] type and are not aimed to give exact a posteriori error bounds. A posteriori estimators with exact bounds for linear functional outputs have been introduced by Paraschivoiu et al. [15]. As pointed out by the authors their focus is not on mesh refinement and the methods need also more computational effort.

In the present work we extend our approach to nonlinear problems. For nonlinear problems approximately the linearized dual problem at the current equilibrium state is utilized for computing the error indicators as will be discussed in section 2. The loading of the dual problem is the same as for linear problems and can be chosen in analogy to the influence line / surface concept as worked out in [6]. The presented indicators give in the asymptotic regime as $h \to 0$ also good error estimators. In the pre-asymptotic regime however the coarse mesh must be able to reproduce the characteristic features of the exact solution. For instationary problems as elastoplasticity or dynamics in addition to the spatial discretization errors the time integration errors must be taken into account. In a first step the time integration errors can be neglected as discussed by Rannacher and Suttmeier [18] and in a more general framework by Radovitzky and Ortiz [17]. In the present work we adopt the same approach for our numerical computations. In contrast to our approach the a posteriori estimators based on the global error in the constitutive law give estimates for the total temporal and spatial discretization errors, Ladeveze et al. [10, 12].

The present approach is restricted to equilibrium points with regular tangential operator and is not applicable to singular points like bifurcation points. For the error estimation at singular points we refer to References [9, 7, 21]. Refinement indicators and adaptive algorithms and their application to shell problems can also be found also in Reference [5].

2 Refinement Indicators for Nonlinear Boundary Value Problems

In the following we derive an error estimator for a general materially nonlinear problem in more detail. The equilibrium equations for geometrically linear problems are given as:

$$
\begin{aligned}
&div\boldsymbol{\sigma}(\boldsymbol{u}) + \lambda\boldsymbol{p} = \boldsymbol{0} \quad \text{on} \quad \Omega \\
&\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n} = \lambda\boldsymbol{f} \quad \text{on} \quad \Gamma_N \qquad \boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D
\end{aligned} \tag{1}
$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{u}$ is the displacement vector, λ is a scalar loading parameter, $\boldsymbol{p}$ is the body force vector, $\boldsymbol{f}$ is the traction prescribed on the Neumann boundary Γ_N, $\boldsymbol{n}$ is the unit normal vector to Γ_N and Γ_D is the Dirichlet boundary. The strain-stress mapping is nonlinear and will be specified later. The weak form of the equilibrium can be derived straightforwardly from equation (1).

$$
\int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v}) dx = \lambda \Big\{ \int_\Omega \boldsymbol{p} \cdot \boldsymbol{v} dx + \int_{\Gamma_N} \boldsymbol{f} \cdot \boldsymbol{v} ds \Big\} \qquad \forall \boldsymbol{v} \in V \tag{2}
$$

We assume the regularity of the solution $\boldsymbol{u}$ so that V is the usual Sobolev space H^1. However, for particular stress-strain relationships discontinuous displacements or localization of the strains

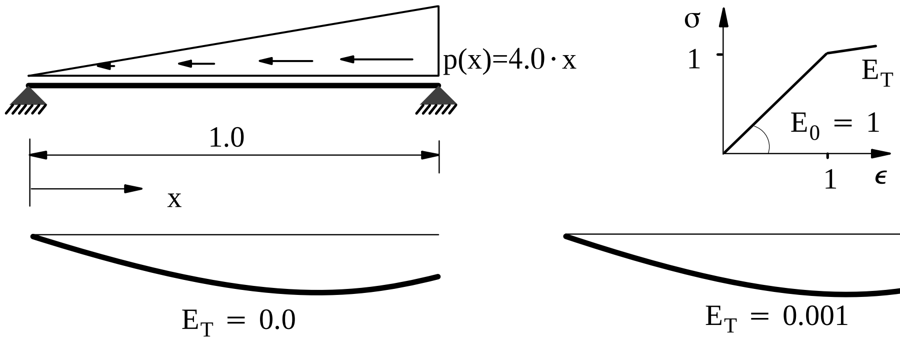


Figure 1: Influence of the tangent E_T on the displacements

is possible and usually a regularization technique must be applied. The simple example of a bar in Fig. 1 shows the occurrence of such discontinuous displacements in dependence of the material behavior. The finite element approximation $\boldsymbol{u}^h$ is computed based on the weak form eq.(2) by choosing $\boldsymbol{v}^h \in V^h$ as a test function.

$$\Big(\boldsymbol{\sigma}(\boldsymbol{u}^h), \boldsymbol{\epsilon}(\boldsymbol{v}^h)\Big) = \lambda\Big\{(\boldsymbol{p}, \boldsymbol{v}^h) + (\boldsymbol{f}, \boldsymbol{v}^h)_{\Gamma_N}\Big\} \qquad \forall \boldsymbol{v}^h \in V^h \subset V \tag{3}$$

Choosing the test function $\boldsymbol{v}$ in equation (2) as $\boldsymbol{v}^h$ and subtracting from equation (3) gives the Galerkin orthogonality or in other terms the best approximation property of the finite element method

$$\Big(\boldsymbol{\sigma}(\boldsymbol{u}) - \boldsymbol{\sigma}(\boldsymbol{u}^h), \boldsymbol{\epsilon}(\boldsymbol{v}^h)\Big) = 0 \qquad \forall \boldsymbol{v}^h \in V \tag{4}$$

Further integration by parts of the internal virtual work of the stress errors gives a boundary value problem for error computation:

$$\Big(\boldsymbol{\sigma}(\boldsymbol{u}) - \boldsymbol{\sigma}(\boldsymbol{u}^h), \boldsymbol{\epsilon}(\boldsymbol{v})\Big) = \sum_{K=1}^{NEL}\Big\{(\boldsymbol{R}, \boldsymbol{v})_{\Omega_K} + (\boldsymbol{J}, \boldsymbol{v})_{\Gamma_K}\Big\} \qquad \forall \boldsymbol{v} \in V \tag{5}$$

Here, NEL is the total number of elements, Ω_K is the domain and Γ_K are the edges of a finite element K. The first term on the right hand side is the virtual work of the element internal residuals $\boldsymbol{R}$

$$div\boldsymbol{\sigma}(\boldsymbol{u}^h) + \lambda\boldsymbol{p} = \boldsymbol{R} \quad \text{on} \quad \Omega_K \tag{6}$$

The virtual work of the jump terms on the internal element edges and on the Neumann boundary is represented by the second term

$$\begin{aligned} &\frac{1}{2}\Big((\boldsymbol{\sigma}(\boldsymbol{u}^h)\cdot\boldsymbol{n})|_K + (\boldsymbol{\sigma}(\boldsymbol{u}^h)\cdot\boldsymbol{n})|_{K^*}\Big) = \boldsymbol{J} \qquad \text{on} \quad \Gamma_K \\ &\boldsymbol{\sigma}(\boldsymbol{u}^h)\boldsymbol{n} - \lambda\boldsymbol{f} = \boldsymbol{J} \qquad \text{on} \quad \Gamma_K \subset \Gamma_N \end{aligned} \tag{7}$$

where $\boldsymbol{n}$ is the normal vector to the joint edge Γ_K of the elements K and K^* or respectively to the boundary Γ_N. The jumps are split by the factor $\frac{1}{2}$ into the two neighboring elements.

Instead of solving the new nonlinear problem eq. (5) the discretization errors can be estimated by a sequence of local problems as motivated by the Galerkin orthogonality. For example

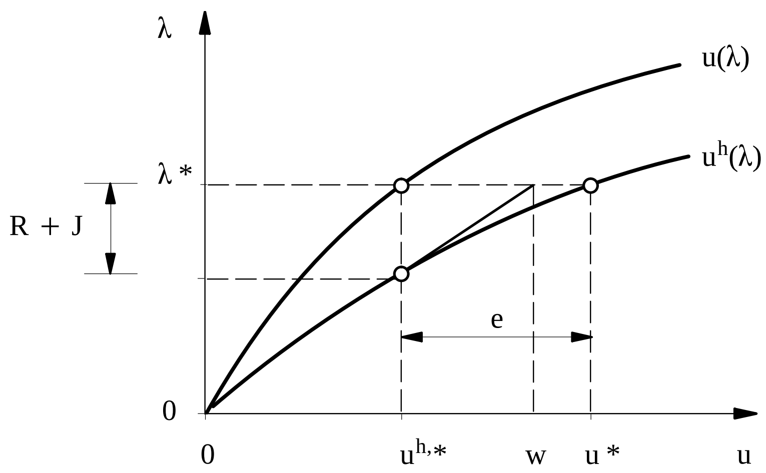


Figure 2: One dimensional example for linearization

local Neumann problems have been utilized by Brink and Stein [4] for finite elasticity and by Galimard et al. [10] for elastoplasticity. Further variationally motivated smoothening based error estimators have been also considered [23, 16]. It should be noted however that always a linearization of the nonlinear equation (5) is necessary. Therefore the nonlinear stress strain relationship is expanded at the finite element solution $\boldsymbol{u}^h$ in a Taylor series.

$$\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{u}) &= \boldsymbol{\sigma}(\boldsymbol{u}^h) + \frac{\partial \boldsymbol{\sigma}(\boldsymbol{u}^h)}{\partial \boldsymbol{\epsilon}} : \boldsymbol{\epsilon}(\boldsymbol{u} - \boldsymbol{u}^h) + \frac{1}{2}\frac{\partial^2 \boldsymbol{\sigma}(\boldsymbol{u}^h)}{\partial \boldsymbol{\epsilon}^2} : \boldsymbol{\epsilon}(\boldsymbol{u} - \boldsymbol{u}^h)^2 \\ &= \boldsymbol{\sigma}(\boldsymbol{u}^h) + \boldsymbol{C}^n : \boldsymbol{\epsilon}(\boldsymbol{u} - \boldsymbol{u}^h) + \cdots \end{aligned} \tag{8}$$

Here $\boldsymbol{C}^n$ is the constitutive tangent of the current state. Provided that the finite element solution $\boldsymbol{u}^h$ is a sufficient approximation for the exact solution $\boldsymbol{u}$, the higher order terms in the series can be neglected. Inserting the series expansion into equation (5) and omitting the higher order terms yields to a linear equation for the discretization errors.

$$\Big(\boldsymbol{C}^n : \boldsymbol{\epsilon}(\boldsymbol{w} - \boldsymbol{u}^h), \boldsymbol{\epsilon}(\boldsymbol{v})\Big) = \sum_{K=1}^{NEL} \Big\{(\boldsymbol{R}, \boldsymbol{v})_{\Omega_K} + (\boldsymbol{J}, \boldsymbol{v})_{\Gamma_K}\Big\} \qquad \forall \boldsymbol{v} \in V \tag{9}$$

Accordingly, the discretization errors $\boldsymbol{u} - \boldsymbol{u}^h$ of nonlinear problems can be estimated in a first approximation by the linearized problem utilizing the displacement field $\boldsymbol{w}$. The constitutive tangent $\boldsymbol{C}^n$ defines the tangential stiffness matrix and is therefore in general available within a finite element computation. A one dimensional interpretation of the equation (9) is given in Fig. 2.

For the error estimation of a selected variable in addition to equation (9) a new problem also called the dual problem has to be defined.

$$\begin{aligned} &div\Big(\boldsymbol{C}^n : \boldsymbol{\epsilon}(\boldsymbol{z})\Big) + \boldsymbol{d} = \boldsymbol{0} \quad \text{on} \quad \Omega \\ &\Big(\boldsymbol{C}^n : \boldsymbol{\epsilon}(\boldsymbol{z})\Big) \cdot \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma_N \qquad \boldsymbol{z} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D \end{aligned} \tag{10}$$

The loading $\boldsymbol{d}$ of the dual problem is specified according to the examined variable. For example for single displacements the loading $\boldsymbol{d}$ of the dual problem consists of a point load or a Dirac

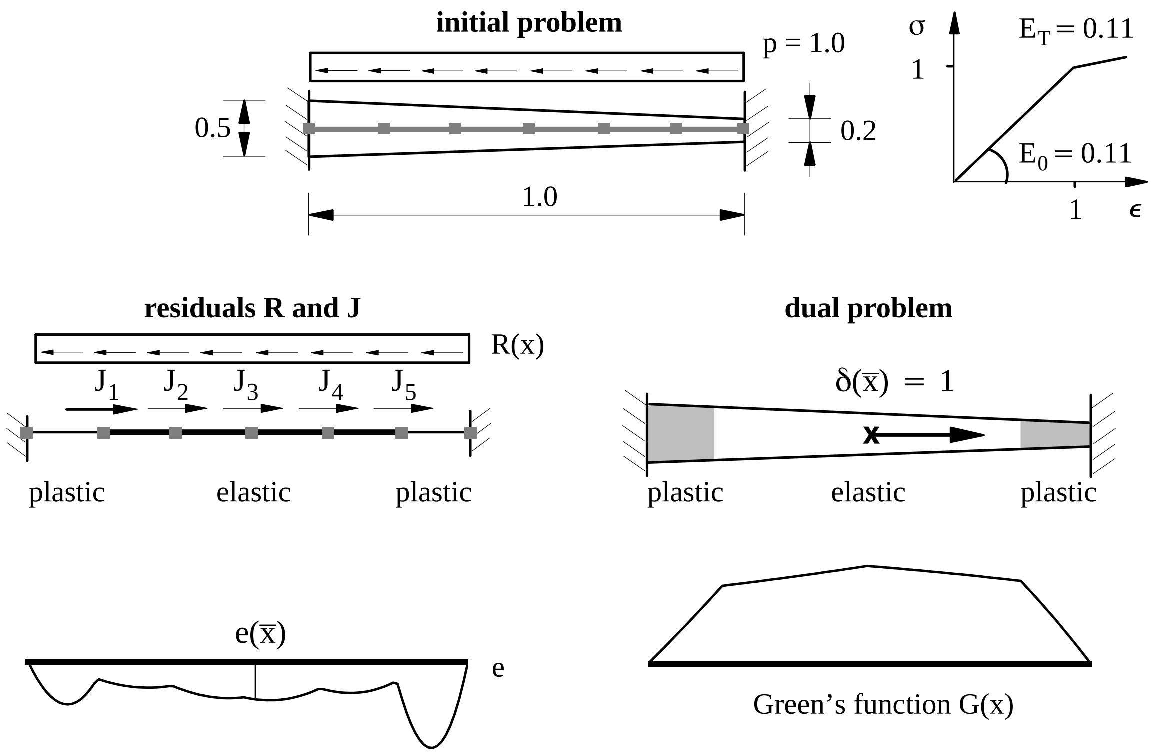


Figure 3: Clamped bar with varying thickness

delta function. It should be noted that for second order differential equations and in two or three dimensions the internal energy of the structure loaded by point loads is infinite [2]. In order to circumvent the possible difficulties we apply a distributed load in a small region ω containing the point of interest $\overline{\boldsymbol{x}}$. For example the errors in the displacements in the direction of the second base vector can be controlled with:

$$\boldsymbol{d} = 0 \cdot \boldsymbol{i}_1 + \overline{f}_2 \cdot \boldsymbol{i}_2 + 0 \cdot \boldsymbol{i}_3 \qquad \overline{f}_2 = \begin{cases} 1 & \text{for} \quad \omega \cap \Omega \\ 0 & \text{for} \quad \overline{\omega \cap \Omega} \end{cases} \tag{11}$$

where $\boldsymbol{i}_1$, $\boldsymbol{i}_2$ and $\boldsymbol{i}_3$ are the cartesian base vectors. Applying the reciprocal theorem of Betti and Rayleigh to the linearized equation (9) and the dual problem (10) yields

$$(\boldsymbol{d}, \boldsymbol{e}) = \Big(\boldsymbol{C}^n : \boldsymbol{\epsilon}(\boldsymbol{w} - \boldsymbol{u}^h), \boldsymbol{\epsilon}(\boldsymbol{z})\Big) = \sum_{K=1}^{NEL} \Big\{(\boldsymbol{R}, \boldsymbol{z})_{\Omega_K} + (\boldsymbol{J}, \boldsymbol{z})_{\Gamma_K}\Big\} \qquad \forall \boldsymbol{v} \in V \tag{12}$$

with $\boldsymbol{e} = \boldsymbol{w} - \boldsymbol{u}^h$. The term on the left hand side is the integrated error in the second component of the displacement vector over the small region ω.

$$(\boldsymbol{d}, \boldsymbol{e}) = \int_\omega e_2 \quad dx \tag{13}$$

The mean value of the displacement error in the subdomain ω can be computed straightforwardly. The derived equations are illustrated with the simple one dimensional example in Figure 3. The bar with varying thickness is loaded by a uniform load and clamped at both ends. The bar is discretized with six linear elements. At the considered load level the first and last element are in the plastic regime. The finite element solution inserted in the equilibrium equations yield the element internal residuals $\boldsymbol{R}$ and the jump residual $\boldsymbol{J}$. As pointed out for error estimation the linearized problem at the finite element solution $\boldsymbol{u}^h$ is considered. For the present example linearization means freezing the elastic and plastic regions at the state $\boldsymbol{u}^h$. Subsequently the local or global errors for the linearized problem can be estimated in a similar way as for the linear problems. For example the local displacement errors can be estimated after inserting the related Green's function in equation 12.

For general problems and control variables the exact solution of the linearized dual problem $\boldsymbol{z}$ is unknown, therefore we utilize the Galerkin orthogonality eq. (4) to introduce the interpolant $I\boldsymbol{z}$.

$$(\boldsymbol{d}, \boldsymbol{e}) = \sum_{K=1}^{NEL} \{(\boldsymbol{R}, \boldsymbol{z} - I\boldsymbol{z})_{\Omega_K} + (\boldsymbol{J}, \boldsymbol{z} - I\boldsymbol{z})_{\Gamma_K}\} \tag{14}$$

The inner products can be separated by the Cauchy-Schwarz inequality

$$|(\boldsymbol{d}, \boldsymbol{e})| \leq \sum_{K=1}^{NEL} \left\{ \|\boldsymbol{R}\|_{0,\Omega_K} \|\boldsymbol{z} - I\boldsymbol{z}\|_{0,\Omega_K} + \|\boldsymbol{J}\|_{0,\Gamma_K} \|\boldsymbol{z} - I\boldsymbol{z}\|_{0,\Gamma_K} \right\} \tag{15}$$

The interpolation errors of the dual solution are bounded by the higher derivatives of the dual solution using the interpolation estimates

$$\|\boldsymbol{z} - I\boldsymbol{z}\|_{0,\Omega_K} \leq C_1 h_K^2 |\boldsymbol{z}|_{2,\Omega_K} \qquad \|\boldsymbol{z} - I\boldsymbol{z}\|_{0,\Gamma_K} \leq C_2 \sqrt{{h_K}^3} |\boldsymbol{z}|_{2,\Omega_K} \tag{16}$$

Here h_K is a characteristic element length and C_1 and C_2 are interpolation constants. The interpolation estimates inserted in equation (15) leads to

$$|(\boldsymbol{d}, \boldsymbol{e})| \leq \sum_{K=1}^{NEL} \{C_1 h_K^2 \|\boldsymbol{R}\|_{0,\Omega_K} |\boldsymbol{z}|_{2,\Omega_K} + C_2 \sqrt{h_K^3} \|\boldsymbol{J}\|_{0,\Gamma_K} |\boldsymbol{z}|_{2,\Omega_K}\} \tag{17}$$

The second order derivatives of the exact solution are unknown. As discussed by Eriksson and Johnson [8] or Rannacher and Suttmeier [18], it is possible to replace the exact second order derivatives approximately by their finite element approximation. Different techniques have been devised to estimate the second order derivatives out of the discontinuous first order derivatives. These schemes are closely related to smoothing procedures as e.g. discussed by [21] and assume the existence of so called superconvergent points [22]. The stress or strain smoothing can be however introduced much earlier in the above derivation so that the computation of the residuals and the interpolation constants C_1 and C_2 can be avoided. We start with the first integral term in equation (12) and introduce the finite element strains of the dual solution $\boldsymbol{\epsilon}(\boldsymbol{z}^h)$ using the Galerkin orthogonality.

$$(\boldsymbol{d}, \boldsymbol{e}) = \Big(\boldsymbol{C}^n : \boldsymbol{\epsilon}(\boldsymbol{w} - \boldsymbol{u}^h), \boldsymbol{\epsilon}(\boldsymbol{z} - \boldsymbol{z}^h)\Big) = \Big(\boldsymbol{\sigma}(\boldsymbol{w}) - \boldsymbol{\sigma}(\boldsymbol{u}^h), \boldsymbol{\epsilon}(\boldsymbol{z}) - \boldsymbol{\epsilon}(\boldsymbol{z}^h)\Big) \tag{18}$$

For error estimation the stresses $\boldsymbol{\sigma}(\boldsymbol{w})$ of the linearized problem and the strains $\boldsymbol{\epsilon}(\boldsymbol{z})$ of the linear dual problem are replaced by postprocessed values $\boldsymbol{\sigma}^*(\boldsymbol{u}^h)$ and $\boldsymbol{\epsilon}^*(\boldsymbol{z}^h)$. The postprocessed values can be evaluation by nodal averaging [24] or by superconvergent patch recovery [25].

The smoothing procedure applied to the exact values and the integral elementwise evaluated yields a refinement indicator.

$$(\boldsymbol{d},\boldsymbol{e}) \approx \sum_{K=1}^{NEL} \Big(\boldsymbol{\sigma}^*(\boldsymbol{u}^h) - \boldsymbol{\sigma}(\boldsymbol{u}^h), \boldsymbol{\epsilon}^*(\boldsymbol{z}^h) - \boldsymbol{\epsilon}(\boldsymbol{z}^h)\Big)_{\Omega_K} \tag{19}$$

This multiplicative procedure for estimating the errors in the inner product $(\boldsymbol{d},\boldsymbol{e})$ has an illustrative interpretation: The first term on the right hand side are the errors of the initial problem and the second term representing the dual problem serves as weighting function and filters out the influence of the overall stress errors over the error in the variable of interest. By an appropriate choice of the dual loading $\boldsymbol{d}$ different refinement indicators can be generated. Furthermore the methods already used in structural mechanics for computation of influence lines / surfaces can be used to find the dual problem. For details we refer to our previous work for linear problems [6].

Remark: For non-selfadjoint boundary value problems resulting for example from a nonassociated flow rule. The dual problem has a different form as in equation (10).

$$\begin{aligned} &div\Big(\boldsymbol{C}^{nT} : \boldsymbol{\epsilon}(\boldsymbol{z})\Big) + \boldsymbol{d} = \boldsymbol{0} \quad \text{on} \quad \Omega \\ &\Big(\boldsymbol{C}^{nT} : \boldsymbol{\epsilon}(\boldsymbol{z})\Big) \cdot \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma_N \quad \boldsymbol{z} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D \end{aligned} \tag{20}$$

with $(\boldsymbol{C}^n_{ijkl})^T = \boldsymbol{C}^n_{klij}$. Using this dual problem and the initial problem as in equation (9) the equivalent to the theorem of Betti and Rayleigh can be derived straightforwardly.

$$\begin{aligned} (\boldsymbol{w} - \boldsymbol{u}^h, \boldsymbol{d}) &= \Big(\boldsymbol{w} - \boldsymbol{u}^h, -div\Big(\boldsymbol{C}^{nT} : \boldsymbol{\epsilon}(\boldsymbol{z})\Big)\Big) = \Big(\boldsymbol{\epsilon}(\boldsymbol{w} - \boldsymbol{u}^h), \boldsymbol{C}^{nT} : \boldsymbol{\epsilon}(\boldsymbol{z})\Big) \\ &= \Big(\boldsymbol{C}^n : \boldsymbol{\epsilon}(\boldsymbol{w} - \boldsymbol{u}^h), \boldsymbol{\epsilon}(\boldsymbol{z})\Big) = \sum_{K=1}^{NEL} \Big\{(\boldsymbol{R},\boldsymbol{z})_{\Omega_K} + (\boldsymbol{J},\boldsymbol{z})_{\Gamma_K}\Big\} \end{aligned} \tag{21}$$

3 Refinement Indicators for Elastoplasticity

The constitutive equations for the elastoplasticity are only given in rate form and have to be integrated over a sequence of discrete time steps. Within each time step the material is simply nonlinear and has to be assumed to be path-independent. Therefore the estimators derived in section 2 for materially nonlinear problems can be applied in an incremental sense to the boundary value problems of elastoplasticity. In the following we briefly recall the small strain elastoplasticity with associative flow rule and isotropic hardening and subsequently discuss the a posteriori error estimation. For given stresses $\boldsymbol{\sigma}_{n-1}$ and load level λ_{n-1} at the time t_{n-1} the stresses $\boldsymbol{\sigma}_n$ at the time t_n are computed from the equilibrium equations.

$$\Big(\Delta\boldsymbol{\sigma}_n, \boldsymbol{\epsilon}(\boldsymbol{v})\Big) = \Delta\lambda_n\Big\{(\boldsymbol{p},\boldsymbol{v}) + (\boldsymbol{f},\boldsymbol{v})_{\Gamma_N}\Big\} \quad \forall \quad \boldsymbol{v} \in V \tag{22}$$

with

$$\Delta\boldsymbol{\sigma}_n = \boldsymbol{\sigma}_n - \boldsymbol{\sigma}_{n-1} \qquad \Delta\lambda_n = \lambda_n - \lambda_{n-1} \tag{23}$$

The related finite element approximation satisfies the following equation:

$$\Big(\Delta\boldsymbol{\sigma}_n^h, \boldsymbol{\epsilon}(\boldsymbol{v}^h)\Big) = \Delta\lambda_n \Big\{(\boldsymbol{p}, \boldsymbol{v}^h) + (\boldsymbol{f}, \boldsymbol{v}^h)_{\Gamma_N}\Big\} \quad \forall \quad \boldsymbol{v}^h \in V^h \subset V \tag{24}$$

As for the stationary problem, the partial integration of the errors in stresses leads to a boundary value problem for computing the stress errors.

$$\Big(\Delta\boldsymbol{\sigma}_n - \Delta\boldsymbol{\sigma}_n^h, \boldsymbol{\epsilon}(\boldsymbol{v})\Big) = \sum_{K=1}^{NEL} \Big\{(\Delta\boldsymbol{R}_n, \boldsymbol{v})_{\Omega_K} + (\Delta\boldsymbol{J}_n, \boldsymbol{v})_{\Gamma_K}\Big\} \qquad \forall \boldsymbol{v} \in V \tag{25}$$

The residuals $\Delta\boldsymbol{R}_n$ and $\Delta\boldsymbol{J}_n$ are computed with the finite element stresses $\Delta\boldsymbol{\sigma}_n^h$ and the load increment $\Delta\lambda_n$. For error estimation we consider again the linearized problem, now for the incremental equations.

$$\Big(\frac{\partial\Delta\boldsymbol{\sigma}_n}{\partial\Delta\boldsymbol{\epsilon}_n} : \boldsymbol{\epsilon}(\Delta\boldsymbol{w}_n - \Delta\boldsymbol{u}_n^h), \boldsymbol{\epsilon}(\boldsymbol{v})\Big) = \sum_{K=1}^{NEL} \Big\{(\Delta\boldsymbol{R}_n, \boldsymbol{v})_{\Omega_K} + (\Delta\boldsymbol{J}_n, \boldsymbol{v})_{\Gamma_K}\Big\} \quad \forall \boldsymbol{v} \in V \tag{26}$$

To evaluate the constitutive tangent the stress strain relationship for elastoplasticity must be specified. The strains $\boldsymbol{\epsilon}$ are splitted into an elastic part $\boldsymbol{\epsilon}^e$ and into a plastic part $\boldsymbol{\epsilon}^p$ so that the following relation for the stresses $\boldsymbol{\sigma}$ holds

$$\boldsymbol{\sigma} = \boldsymbol{C} : (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^p) \tag{27}$$

where $\boldsymbol{C}$ is the elastic constitutive tensor. The plastic strains $\boldsymbol{\epsilon}^p$ and the hardening parameter ξ are given in rate form as

$$\dot{\boldsymbol{\epsilon}}^p = \dot{\gamma}\boldsymbol{r} \qquad \dot{\xi} = \dot{\gamma} h \qquad \text{with} \quad \boldsymbol{r} = \frac{\partial F}{\partial \boldsymbol{\sigma}} \quad \text{and} \quad h = \frac{\partial F}{\partial \xi} \tag{28}$$

where F is the yield condition and $\dot{\gamma}$ is the plastic multiplier. The occurrence of plasticity and the loading and unloading conditions are expressed by the Kuhn-Tucker complementarity conditions.

$$\dot{\gamma} \geq 0 \qquad F(\boldsymbol{\sigma}, \xi) \leq 0 \qquad \dot{\gamma} F(\boldsymbol{\sigma}, \xi) = 0 \tag{29}$$

The multiplier $\dot{\gamma}$ can be computed out of the consistency condition $\dot{F} = 0$ as

$$\dot{\gamma} = \frac{\boldsymbol{r} : \boldsymbol{C} : \dot{\boldsymbol{\epsilon}}}{A + \boldsymbol{r} : \boldsymbol{C} : \boldsymbol{r}} \qquad \text{with} \quad A = -h\frac{\dot{\xi}}{\dot{\gamma}} \tag{30}$$

To obtain the strains and subsequently the stresses the rate equations (28) must be integrated. Applying the implicit Euler scheme leads to

$$\begin{aligned} \boldsymbol{\epsilon}_n^p &= \boldsymbol{\epsilon}_{n-1}^p + \Delta\gamma_n \boldsymbol{r}_n \\ \xi_n &= \xi_{n-1} + \Delta\gamma_n h_n \end{aligned} \tag{31}$$

with

$$\Delta\gamma_n = \frac{\boldsymbol{r}_n : \boldsymbol{C} : \Delta\boldsymbol{\epsilon}_n}{A + \boldsymbol{r}_n : \boldsymbol{C} : \boldsymbol{r}_n} \tag{32}$$

The equations (31) and (32) and further the discrete version of the Kuhn-Tucker conditions eq.(29) establish a discrete constrained optimization problem for computing the plastic strain $\boldsymbol{\epsilon}_n^p$ and the hardening parameter ξ_n. The equation (31) for plastic strain $\boldsymbol{\epsilon}_n^p$ inserted into equation (27) gives a nonlinear not path dependent equation for computing the stresses $\boldsymbol{\sigma}_n$. In general the nonlinear equations are solved in two steps consisting of an elastic predictor and a plastic corrector step. At first an elastic trial state with $\boldsymbol{\sigma}_n^{trial}$ is computed.

$$\boldsymbol{\sigma}_n^{trial} = \boldsymbol{C} : (\boldsymbol{\epsilon}_n - \boldsymbol{\epsilon}_{n-1}^p) \tag{33}$$

Subsequently the stress is reduced with a return mapping algorithm if plastic deformation occurs.

$$\boldsymbol{\sigma}_n = \boldsymbol{\sigma}_n^{trial} - \Delta\gamma_n \boldsymbol{C} : \boldsymbol{r}_n \tag{34}$$

Further specific features of the integration algorithms are for the a posteriori error estimation not relevant, therefore we refer for details to standard literature as Simo and Hughes [19]. The equations for the stress evaluation are now specified and the constitutive tangent $\boldsymbol{C}^n$ can be derived out of equation (34). For incremental strains leading to a plastic deformation the material tangent, needed for equation (26) has the following form

$$\boldsymbol{C}^{ep} = \frac{\partial \Delta\boldsymbol{\sigma}_n}{\partial \Delta\boldsymbol{\epsilon}_n} = \boldsymbol{H} - \frac{\boldsymbol{H} : \boldsymbol{r}_n \otimes \boldsymbol{r}_n : \boldsymbol{H}}{A + \boldsymbol{r}_n : \boldsymbol{H} : \boldsymbol{r}_n} \tag{35}$$

with

$$\boldsymbol{H} = \Big(\boldsymbol{I} + \Delta\gamma_n \boldsymbol{C} : \frac{\partial \boldsymbol{r}_n}{\partial \boldsymbol{\sigma}}\Big)^{-1} \boldsymbol{C} \tag{36}$$

For incremental deformations without a plastic strain part the constitutive relation for elasticity is still valid. Using the derived constitutive tangent and the equation (26) the linearized problem is utilized for error estimation.

$$\Big(\boldsymbol{C}^n : \boldsymbol{\epsilon}(\Delta\boldsymbol{e}_n), \boldsymbol{\epsilon}(\boldsymbol{v})\Big) = \sum_{K=1}^{NEL} \Big\{(\Delta\boldsymbol{R}_n, \boldsymbol{v})_{\Omega_K} + (\Delta\boldsymbol{J}_n, \boldsymbol{v})_{\Gamma_N}\Big\} \qquad \forall \boldsymbol{v} \in V \tag{37}$$

with $\Delta\boldsymbol{e}_n = \Delta\boldsymbol{w}_n - \Delta\boldsymbol{u}_n^h$. The constitutive tensor $\boldsymbol{C}^n$ depends on the current state of the considered material point.

$$\boldsymbol{C}^n = \begin{cases} \boldsymbol{C} & \text{for elastic points} \\ \boldsymbol{C}^{ep} & \text{for plastic points} \end{cases} \tag{38}$$

The further steps of the derivation are similar to the general materially nonlinear case of section 2. Especially the dual problem in not dependent on the incremental approach and is evaluated as for the stationary nonlinear problem. The duality based smoothening type estimator corresponding to the estimator in equation (19), has the following form.

$$(\boldsymbol{d}, \Delta\boldsymbol{e}_n) \approx \sum_{K=1}^{NEL} \Big(\Delta\boldsymbol{\sigma}_n^{h^*} - \Delta\boldsymbol{\sigma}_n^h, \boldsymbol{\epsilon}^*(\boldsymbol{z}^h) - \boldsymbol{\epsilon}(\boldsymbol{z}^h)\Big)_{\Omega_K} \tag{39}$$

For error computation only the difference of the stresses at the end and at the beginning of each time step are used. The total error is the sum of the errors of the incremental problems.

$$(\boldsymbol{d}, \boldsymbol{e}) = \sum_{i=1}^{N} (\boldsymbol{d}, \Delta \boldsymbol{e}_i) \tag{40}$$

The present approach can capture in a very easy way the overall behavior of the structure and is therefore especially suited for elastoplastic problems. As a simple local analysis of the acoustic tensor shows, the elastoplastic tensor can become strongly anisotropic depending on the hardening parameter and varies through the whole plastic region. For such a complex structural problem the solution of only local problems cannot give efficient error estimators as already noticed by different authors [14, 3].

4 Adaptive Mesh Refinement

The discretized nonlinear boundary value problems are solved with the usual predictor-corrector algorithm and the Newton-Raphson iteration. The introduced error estimator is integrated in the path following scheme. At equilibrium points the discretization errors are estimated through the linearized problem. If accuracy is not sufficient a new mesh is generated based on the computed element error indicators of equation (19) or (39) respectively. We use for the refinement procedure the absolute value of the error contributed by each element.

$$\rho_K = \left| \Big(\boldsymbol{\sigma}^*(\boldsymbol{u}^h) - \boldsymbol{\sigma}(\boldsymbol{u}^h), \boldsymbol{\epsilon}^*(\boldsymbol{z}^h) - \boldsymbol{\epsilon}(\boldsymbol{z}^h) \Big)_{\Omega_K} \right| \tag{41}$$

As already described the tangential stiffness matrix at the equilibrium point is used for computing the displacements $\boldsymbol{z}^h$ of the linear dual problem. Applying a direct solver in the finite element code the stiffness matrix must be, of course, factorized only once (Fig. 4). The load vector of the dual problem depends on the control variable and can be constructed in analogy to the influence line / surface concept of structural analysis. Based on the dual solution $\boldsymbol{z}^h$ the strain $\boldsymbol{\epsilon}(\boldsymbol{z}^h)$ is computed with the usual linear strain displacement relationship.In the existing finite element programs the stresses $\boldsymbol{\sigma}$ are computed during the elasto-plastic solution procedure and therefore also available for error estimation. Out of the stresses $\boldsymbol{\sigma}$ and strains $\boldsymbol{\epsilon}$ the smoothed values $\boldsymbol{\sigma}^*$ and $\boldsymbol{\epsilon}^*$ are computed by nodal averaging.

Based on the refinement indicator eq. (41) the element lengths for the new mesh are computed in the same way as for linear problems. Therefore we recall only briefly the related procedures for the remeshing strategy; for the details we refer to [5, 6]. The new element lengths are computed using the principle of error equidistribution over the whole mesh. In dependence of the user specified relative error $\bar{\eta}$ the error contribution of each element of the old mesh should be

$$\bar{e}_m = \frac{\bar{\eta}(\boldsymbol{u}^h, \boldsymbol{d})}{NEL} \tag{42}$$

where NEL is the number of elements. Further the following ratio is defined

$$\bar{\xi}_K = \frac{\rho_K}{\bar{e}_m} \tag{43}$$

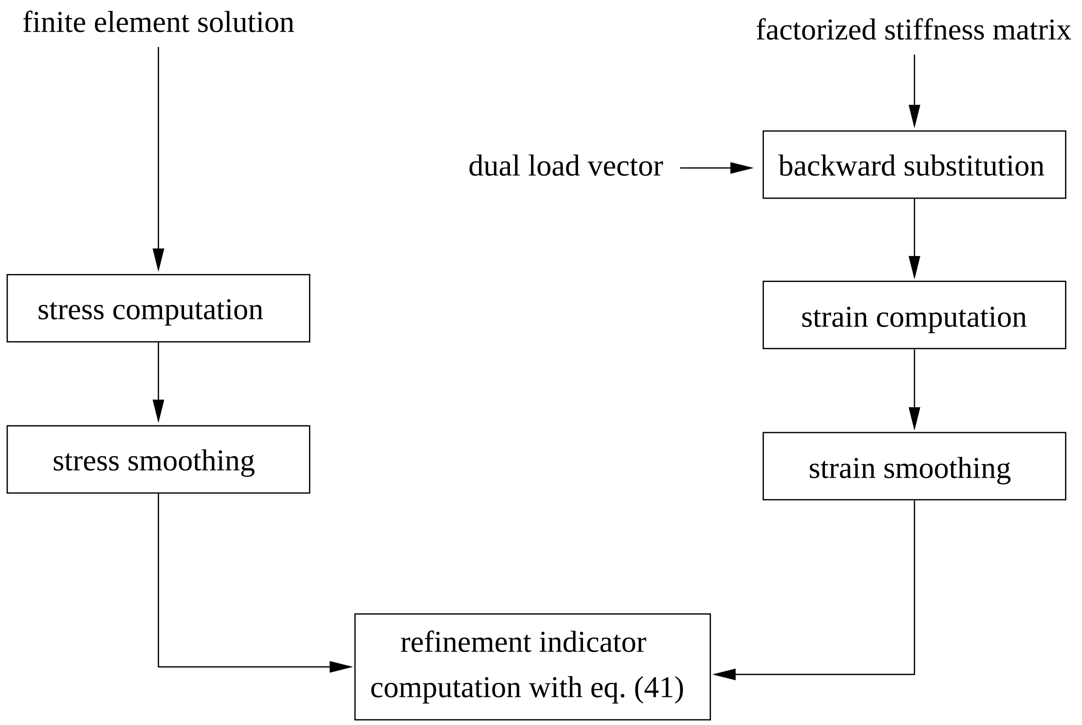


Figure 4: Refinement indicator evaluation

It should be noted that $\bar{\xi}_K$ is not dependent on the length of the dual load vector. Using the convergence rates of a priori estimates for smooth solutions the element lengths for displacement error control is computed as

$$h_{K,new} = \bar{\xi}_K^{-\frac{1}{p+1}} h_{K,old} \tag{44}$$

For stress error control the lower convergence rate should be taken into account.

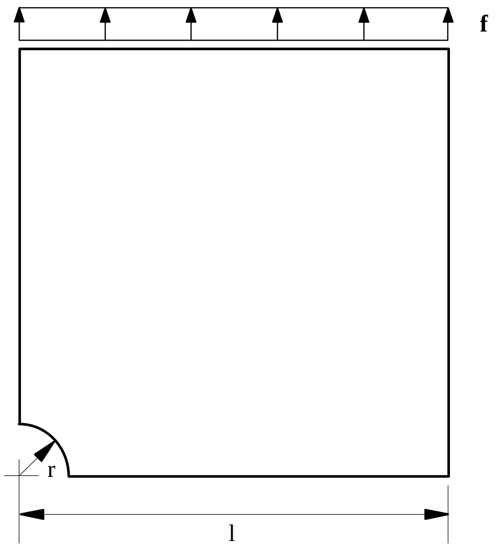
$$h_{K,new} = \bar{\xi}_K^{-\frac{1}{p}} h_{K,old} \tag{45}$$

whereby p is the polynomial degree of the element shape functions. Based on the computed element lengths h_{new} a new mesh is generated. Subsequently the state variables are transfered from the old to the new mesh. For the path dependent elastoplasticity problem the plastic strains and the internal variables at the Gauss points must be transferred. Furthermore to ensure the convergence of the subsequent equilibrium iteration also the nodal displacements should be transferred. The internal variables can be transferred in six steps.

1. Transfer the history variables from the Gauss points to the nodal points by extrapolation.

2. Smooth the discontinuous nodal values by nodal averaging to minimize the amount of the numerical noise.

3. Search for the element of the old mesh which contains the node of the new mesh.



length	l	=	10.00
radius	r	=	1.00
Young's mod	E	=	206900
Poisson's ratio	ν	=	0.29
hardening	H	=	0.0
yield stress	σ_Y	=	450.0
plane strain			

Figure 5: Perforated tension strip (one quarter)

4. Compute the local coordinates of the new node.

5. Interpolate the nodal values with the finite element shape functions.

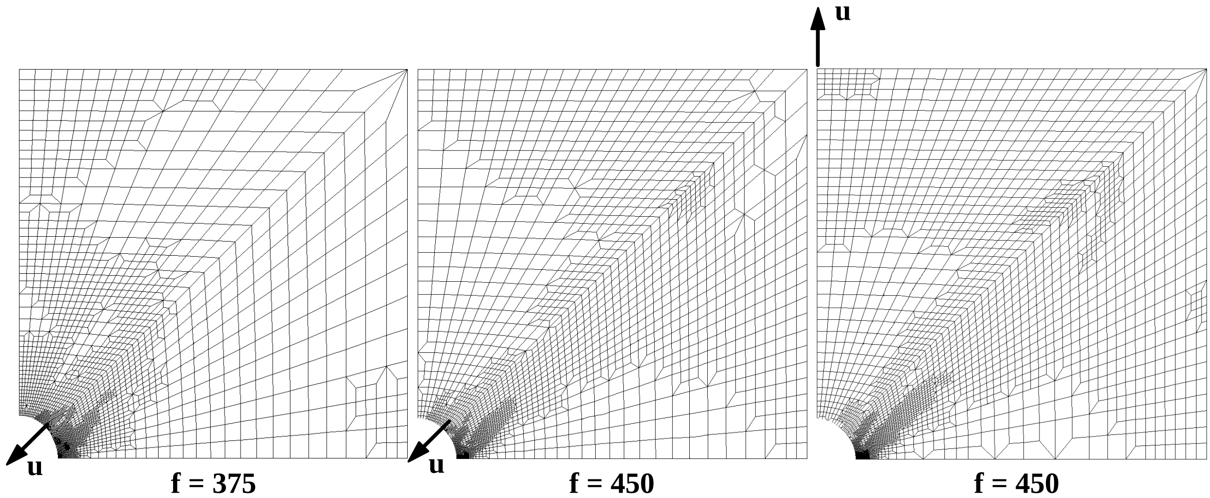
6. Transfer the nodal values of the history variables to the Gauss points.

After the state variables and displacements are transferred the mesh is equilibrated with a corrector iteration and the error is estimated for the new mesh.

5 Examples

The presented refinement indicators enable the mesh refinement with respect to almost arbitrarily defined variables. The indicators rely on the linearized problem and give in the asymptotic regime also good error estimators. For validating our theoretical framework, we consider the well known perforated tension strip example. Due to symmetry only one quarter of the structure is discretized with four noded displacement elements. The geometry and material data are given in Fig. 4. The hardening parameter is set to zero in order to test the method in the extreme case. The linearized dual problem captures the global structural behavior and gives therefore good error indicators for perfect plasticity. The refinement indicators gives in the limit also useful error estimators as the following experiments with the different control points and load levels show.

The total load $\boldsymbol{f}$ of 375 and 450 are applied in one step, in order to simulate the deformation theory by a classical algorithm for J_2 plasticity. The deformation theory type of plasticity describes only a nonlinear material behavior and fits into the framework of chapter 2. We control for example the errors of displacements at different finite element nodes in the elastic and plastic region indicated by arrows in Fig. 6. The quasi exact relative errors η^{ex} are computed by comparison of the current solution with a fine mesh solution. As shown in the tables the



NDOF	u	η	η^{ex}
360	$4.21 \cdot 10e{-}3$	3.50	5.70
1222	$4.41 \cdot 10e{-}3$	0.60	0.90
2426	$4.43 \cdot 10e{-}3$	0.33	0.45
7995	$4.45 \cdot 10e{-}3$	0.1	

NDOF	u	η	η^{ex}
360	$7.90 \cdot 10e{-}3$	9.19	15.8
1333	$8.88 \cdot 10e{-}3$	3.31	3.04
2397	$8.99 \cdot 10e{-}3$	2.05	1.80
8918	$9.15 \cdot 10e{-}3$	0.32	

NDOF	u	η	η^{ex}
360	$2.37 \cdot 10e{-}2$	2.62	4.20
744	$2.42 \cdot 10e{-}2$	1.72	2.07
1224	$4.44 \cdot 10e{-}2$	1.08	0.82
2159	$2.46 \cdot 10e{-}2$	0.65	0.40
17038	$2.47 \cdot 10e{-}2$		

Figure 6: Error control for selected finite element nodes

smoothening based estimator gives excellent results. The estimated and the quasi exact relative errors converge very fast to comparable values. For all controlled nodal values the refinement indicator leads to refinement at the control points and also at the transition zone from elastic to plastic region. The same example is also analyzed by the classical J_2 flow theory. For controlling the errors of the vertical displacement at the center of the upper edge we use the indicator of equation (39). A sharp error tolerance of 0.5% was prescribed. After an initial mesh refinement in the elastic range the load and displacements are scaled to the stage where the plastic range starts. Subsequently the load is applied by displacement control of the center of the upper edge using displacement increments of 0.001. The related load displacement diagram as well as the evolution of the error are shown in Fig. 7. The spreading of the plastic zone is

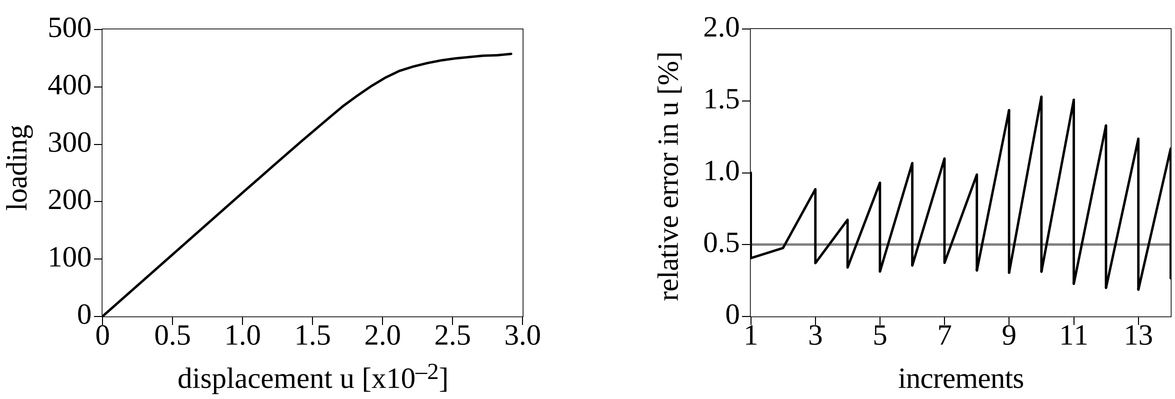


Figure 7: Load displacement diagram and error evolution

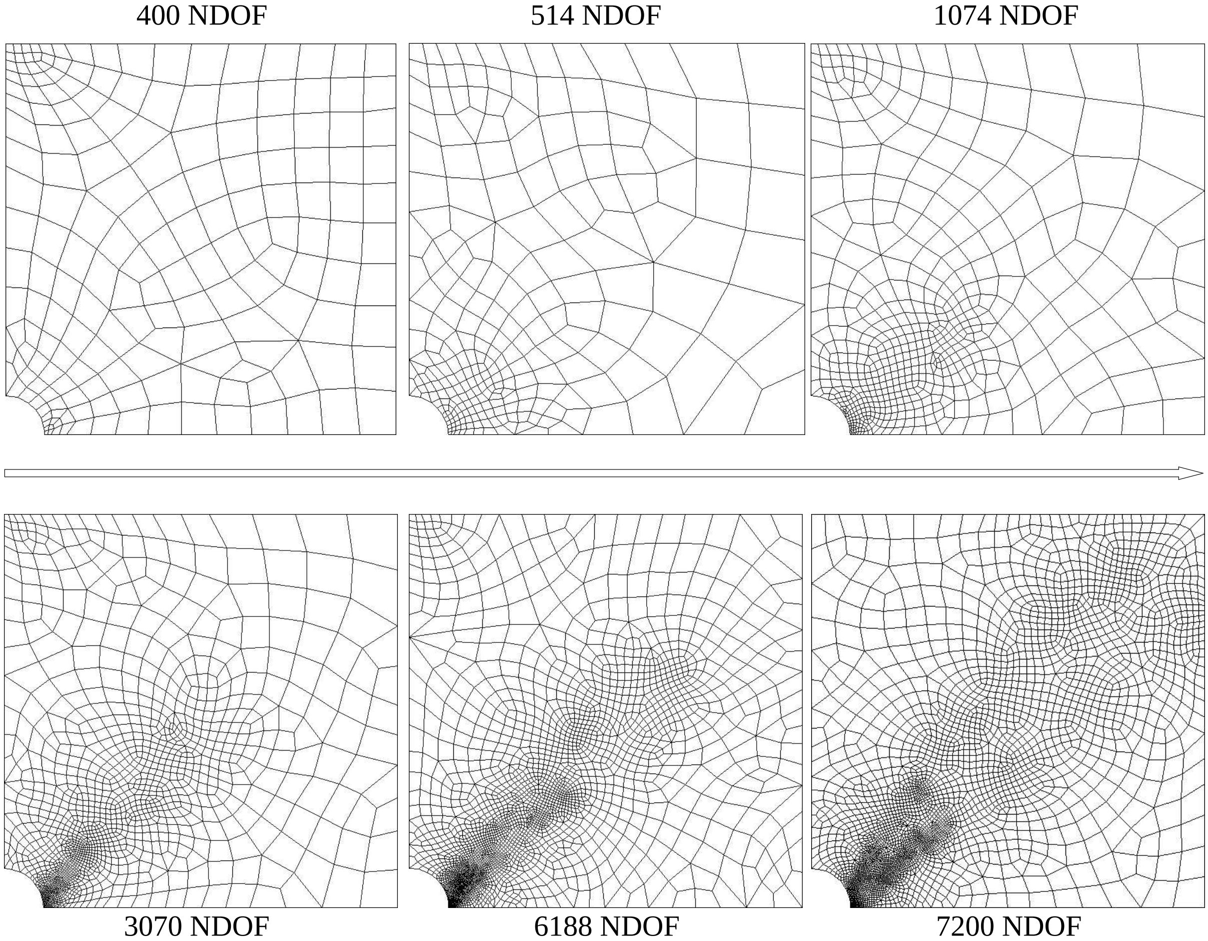


Figure 8: Sequence of refined meshes

captured by the adaptive algorithm with frequent mesh refinement as shown in Fig. 8 . After each mesh refinement the state variables are transferred as described in section 5. As observed often, the subsequent iteration fails to converge without appropriate modifications. Therefore we control at the beginning of the equilibrium iteration the discrete residuals and cut or magnify the displacement increments if the iteration tends to diverge or to cycle. Figure 8 shows the starting mesh and some selected intermediate meshes and the final mesh. As it is evident from the refined meshes the transition zones with high gradients are resolved. To capture the whole loading path 12 mesh refinements were necessary. For the present examples we applied the proposed estimators only for error control of local displacements. However, the refinement indicators can be also used for error control in other variables as local stresses or boundary tractions. For the related refinement indicators only the dual problem must be changed.

6 Conclusion

The cornerstone of the proposed methods is the use of duality techniques or in other notions the reciprocal theorem. The dual problem captures the influence of the element errors to the considered variable and reflects the concept of influence lines and surfaces very well understood in structural mechanics. Furthermore the element contributions are applied as a refinement indicator in connection with an h-adaptive algorithm. As discussed the present approach is not restricted to elastoplasticity and can be simply extended to other problem classes. Importantly, the dual problem is used for appropriate weighting of the separate contributions to the total error in contrast to the frequently used error estimators with only one global strong stability constant. The presented numerical examples show the accuracy and the applicability of the related theoretical framework.

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