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A-Posteriori Error Estimation and Adaptivity for the Finite Element Method Using Duality Principles

The present study introduces the concept of error estimation for locally and globally defined variables in structural mechanics. As a basic component of the formulation a dual problem specifically designed for the variables in mind is defined. The influence of the errors of the entire spatial or temporal domain on the local error of the specific variable is filtered out by the corresponding dual (influence) problem. The duality techniques are related to the principle of Betti-Maxwell and can be used for error estimation with respect to various locally or globally defined variables.

1. Introduction

It is not always sufficient to reduce the reliability of the finite element computation to a global measure, as for example the energy norm, alone. Often the quality of local quantities, e.g. displacements or stresses in a region, are of even more interest. The idea of local a-posteriori error estimation can be traced back already to Tottenham [5] at the beginning of the seventies. Later Johnson [4] and coworkers and Becker and Rannacher [1] introduced the general framework of local error estimation in a mathematical context for all kinds of differential equations based on duality arguments. In chapter 2 we introduce local error estimators for linear elastostatics establishing the dual problem according to the local error quantity. In a straight forward way the extension to nonlinear problems is shown in chapter 3 applying the method to the linearized problem. The estimator is also applied to instationary Prandtl-Reuss plasticity in an incremental sense.

2. Error Estimation for Linear Elastostatics

The principle of virtual work for linear elastostatics is given by

$$\int_\Omega \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{\epsilon}(\boldsymbol{v})dx = \int_\Omega \boldsymbol{p}\boldsymbol{v}dx + \int_{\Gamma_N} \boldsymbol{f}\boldsymbol{v}dx \qquad \forall \boldsymbol{v} \in V(\Omega) := \{\boldsymbol{v} \in H^1 : \boldsymbol{v} = \boldsymbol{0} \quad on \quad \Gamma_D\} \tag{1}$$

mm where $\boldsymbol{\sigma}$ are the stresses, $\boldsymbol{u}$ are the displacements, $\boldsymbol{v}$ are the test functions or the virtual displacements, $\boldsymbol{p}$ are the body forces, $\boldsymbol{f}$ are the tractions prescribed on the force boundary Γ_N and Γ_D is the displacement boundary. In the following we use for equation (1) the abreviated form

$$B(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{p},\boldsymbol{v})_\Omega + (\boldsymbol{f},\boldsymbol{v})_{\Gamma_N} \qquad \forall \boldsymbol{v} \in V \tag{2}$$

Using appropriate finite element shape functions a conforming finite element approximation $\boldsymbol{u}^h$ to the displacement $\boldsymbol{u}$ can be computed in the standard way. The related discretization error $\boldsymbol{e} = \boldsymbol{u} - \boldsymbol{u}^h$ is the solution to the following modified boundary value problem.

$$B(\boldsymbol{e},\boldsymbol{v}) = \sum_{K=1}^{NEL} \{(\boldsymbol{R},\boldsymbol{v})_{\Omega_K} + (\boldsymbol{J},\boldsymbol{v})_{\Gamma_K}\} \qquad \forall \boldsymbol{v} \in V \tag{3}$$

The term on the left hand side partially integrated gives the appropriate expressions for the internal residuals $\boldsymbol{R}$ and jump terms $\boldsymbol{J}$ between adjacent elements. Instead of solving the new problem eq. (3) the discretization errors can be estimated by local considerations as will be discussed subsequently.

For the error estimation for a selected variable in addition to equation (3) a new problem also called the dual problem has to be defined.

$$B(\boldsymbol{z},\boldsymbol{v}) = (\boldsymbol{d},\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in V \tag{4}$$

The loading $\boldsymbol{d}$ of the dual problem is specified according to the examined variable. For example for single displacements the loading $\boldsymbol{d}$ of the dual problem consists of a point load. It should be noted that for second order differential equations and in two or three dimensions the internal energy of the structure loaded by point loads is infinite. In

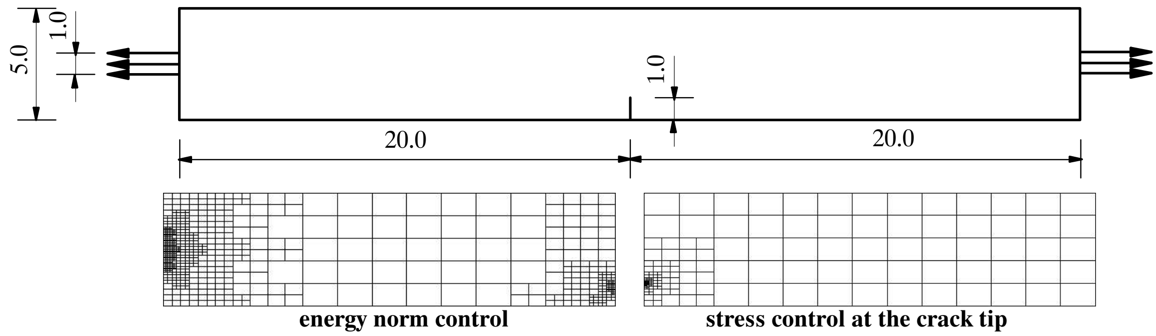

Figure 1: Cracked plate loaded by concentrate forces

order to circumvent the possible diffuculties we apply a distributed load $\overline{\boldsymbol{f}}_i$ in a small region ω containing the point of interest $\overline{\boldsymbol{x}}$. The load vector is equal to one only in the i-th component and otherwise equal to zero e.g. for $i = 2$

$$\boldsymbol{d} = 0 \cdot \boldsymbol{e}_1 + \overline{f}_2 \boldsymbol{e}_2 + 0 \cdot \boldsymbol{e}_3 \qquad \overline{f}_2 = \begin{cases} 1 & \text{for} \quad \omega \cap \Omega \\ 0 & \text{for} \quad \overline{\omega \cap \Omega} \end{cases} \tag{5}$$

Applying the reciprocal theorem of Betti and Rayleigh to equation (3) and the dual problem (4), we obtain the following relation for the error

$$\int_\omega e_2 dx = (\boldsymbol{e}, \boldsymbol{d}) = B(\boldsymbol{z}, \boldsymbol{e}) = B(\boldsymbol{e}, \boldsymbol{z}) = \sum_{K=1}^{NEL} \{(\boldsymbol{R}, \boldsymbol{z} - \boldsymbol{z}^h)_{\Omega_K} + (\boldsymbol{J}, \boldsymbol{z} - \boldsymbol{z}^h)_{\Gamma_K}\} \tag{6}$$

The term on the right hand side is the integrated error over the small region ω. Because the exact function $\boldsymbol{z}$ on the right hand side is unknown the Galerkin orthogonality is utilized to introduce the finite element approximation $\boldsymbol{z}^h$

$$\int_\omega e_2 dx = B(\boldsymbol{e}, \boldsymbol{z} - \boldsymbol{z}^h) = \sum_{K=1}^{NEL} \{(\boldsymbol{R}, \boldsymbol{z} - \boldsymbol{z}^h)_{\Omega_K} + (\boldsymbol{J}, \boldsymbol{z} - \boldsymbol{z}^h)_{\Gamma_K} \tag{7}$$

Further applying the Cauchy-Schwarz inequality the local error is estimated by the product of the errors in the energy norm of the initial and the dual problem.

$$\int_\omega e_2 dx = \sqrt{\sum_{K=1}^{NEL} B(\boldsymbol{e}, \boldsymbol{e})_{\Omega_k} B(\boldsymbol{z} - \boldsymbol{z}^h, \boldsymbol{z} - \boldsymbol{z}^h)_{\Omega_k}} \tag{8}$$

This means that we can apply the classical residual or postprocessing based energy norm estimators to initial and dual problems and compute an estimate for the local error. This multiplicative procedure has also an illustrative interpretation: The second term representing the dual problem serves as weighting function and filters out the influence of the overall residuals over the displacement error of interest.

The cracked plate example in Figure 1 shows the application of the present approach. The ultimate load capacity of the plate depends on the stress intensity factor at the tip of the crack. An analytical solution for the stress intensity factor K_I is

$$K_I = \lim_{rto0} \sqrt{2\Pi r \sigma_{xx}} \tag{9}$$

with σ_{xx} denoting the horizontal stress in the distance r from the crack tip. It is reasonable to specifically adapt the mesh due to the local horizontal stress near the tip which leads to the stress intensity factor of $K_I = 2.66$ which is close to the analytical solution of an infinite plate $K_I = 2.42$. The comparison of the final meshes indicates the advantage of local error estimators refining the mesh only locally in the interested region; there the error is stronger weighted in contrast to the globally averaging energy norm estimators which refines the mesh also close to the applied external load.

3. Refinement Indicators for Nonlinear Problems

For not path dependent problems, like geometric nonlinearities or the deformation theory of plasticity (Hencky-Plasticity), the previous formulations are applied to the linearized problem. This holds for regular points at which

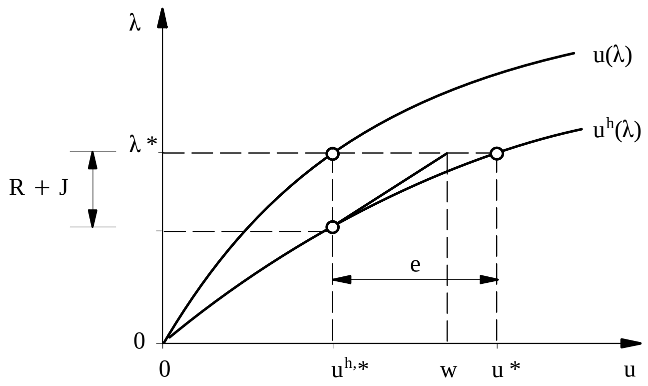

Figure 2: One dimensional example for linearization

the continuous linearized problem is not singular. As in the linear case the differential equation for the discretization errors can be derived by partial integration of the errors in the internal work.

$$\big(\boldsymbol{\sigma}(\boldsymbol{u}) - \boldsymbol{\sigma}(\boldsymbol{u}^h), \boldsymbol{\epsilon}(\boldsymbol{v})\big) = \sum_{K=1}^{NEL} (\boldsymbol{R}, \boldsymbol{v})_{\Omega_K} + (\boldsymbol{J}, \boldsymbol{v})_{\Gamma_K} \qquad \forall v \in V \tag{10}$$

For the Hencky-Plasticity the nonlinear stress strain relationship is expanded in a Taylor series.

$$\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{\sigma}(\boldsymbol{u}^h) + \frac{\partial \boldsymbol{\sigma}(\boldsymbol{u}^h)}{\partial \boldsymbol{\epsilon}} : \boldsymbol{\epsilon}(\boldsymbol{u} - \boldsymbol{u}^h) + \frac{1}{2}\frac{\partial^2 \boldsymbol{\sigma}(\boldsymbol{u}^h)}{\partial \boldsymbol{\epsilon}} \boldsymbol{\epsilon}(\boldsymbol{u} - \boldsymbol{u}^h) = \boldsymbol{\sigma}(\boldsymbol{u}^h) + \boldsymbol{C}^n : \boldsymbol{\epsilon}(\boldsymbol{u} - \boldsymbol{u}^h) + \cdots \tag{11}$$

The material tensor $\boldsymbol{C}^n$ is either the usual elastic or the elastoplastic tensor depending on the state of the material point. Inserting the series expansion into equation (10) and ommiting the higher order terms yields to a linear equation for the discretization errors. The dual problem is defined as in the linear case however with current properties of the problem.

$$\big(\boldsymbol{C}^n : \boldsymbol{\epsilon}(\boldsymbol{z}^h), \boldsymbol{\epsilon}(\boldsymbol{v})\big) = (\boldsymbol{d}, \boldsymbol{v}) \tag{12}$$

Applying the principle of Betti-Maxwell to the initial and the dual problem leads to

$$(\boldsymbol{d}, \boldsymbol{e}) = \big(\boldsymbol{C}^n : \boldsymbol{\epsilon}(\boldsymbol{w} - \boldsymbol{u}^h), \boldsymbol{\epsilon}(\boldsymbol{z})\big) = \sum_{K=1}^{NEL} \{(\boldsymbol{R}, \boldsymbol{z})_{\Omega_K} + (\boldsymbol{J}, \boldsymbol{z})_{\Gamma_K}\} \qquad \forall \boldsymbol{v} \in V \tag{13}$$

Based on this equation and applying the inherent Galerkin orthogonality it is possible to derive a residual based error estimator or a simple smoothing type error estimator.

$$(\boldsymbol{d}, \boldsymbol{e}) = \big(\boldsymbol{C}^n : \boldsymbol{\epsilon}(\boldsymbol{w} - \boldsymbol{u}^h), \boldsymbol{\epsilon}(\boldsymbol{z} - \boldsymbol{z}^h)\big) = \big(\boldsymbol{\sigma}(\boldsymbol{w}) - \boldsymbol{\sigma}(\boldsymbol{u}^h), \boldsymbol{\epsilon}(\boldsymbol{z}) - \boldsymbol{\epsilon}(\boldsymbol{z}^h)\big) \tag{14}$$

For error estimation the stresses $\boldsymbol{\sigma}(\boldsymbol{w})$ of the initial and the strains $\boldsymbol{\epsilon}(\boldsymbol{G})$ of the linear dual problem are replaced by postprocessed values, see the load displacement diagram Figure 2 for notation. For example $\boldsymbol{\sigma}(\boldsymbol{w})$ and $\boldsymbol{\epsilon}(\boldsymbol{G})$ can be computed by simple nodal averaging [6].

For path dependent problems, like in flow theory of plasticity the duality arguments can be applied in an incremental sense. The example of a square elastoplastic perforated strip (Figure 3) demonstrates the effectivity of the current approach. Due to symmetry only one quarter of the structure is analyzed using bilinear quadrilateral finite elements. The error of the vertical displacement at the middle of the top rim is controlled. The load is applied by displacement control with a stepsize of 0.001. The mesh is adapted within each increment to the error tolerance of 0.5% of the top displacement error. The relative error distribution over the increments can be seen in Figure 3 as well as the load displacement curve.

4. Conclusion

The fundamental issue of this study is to extend the error estimation and adaptivity based on duality principles to further problems in structural mechanics. The dual problem captures the influence of the residuals to the considered variable and reflects the concept of influence lines and surfaces very well understood in structural mechanics.

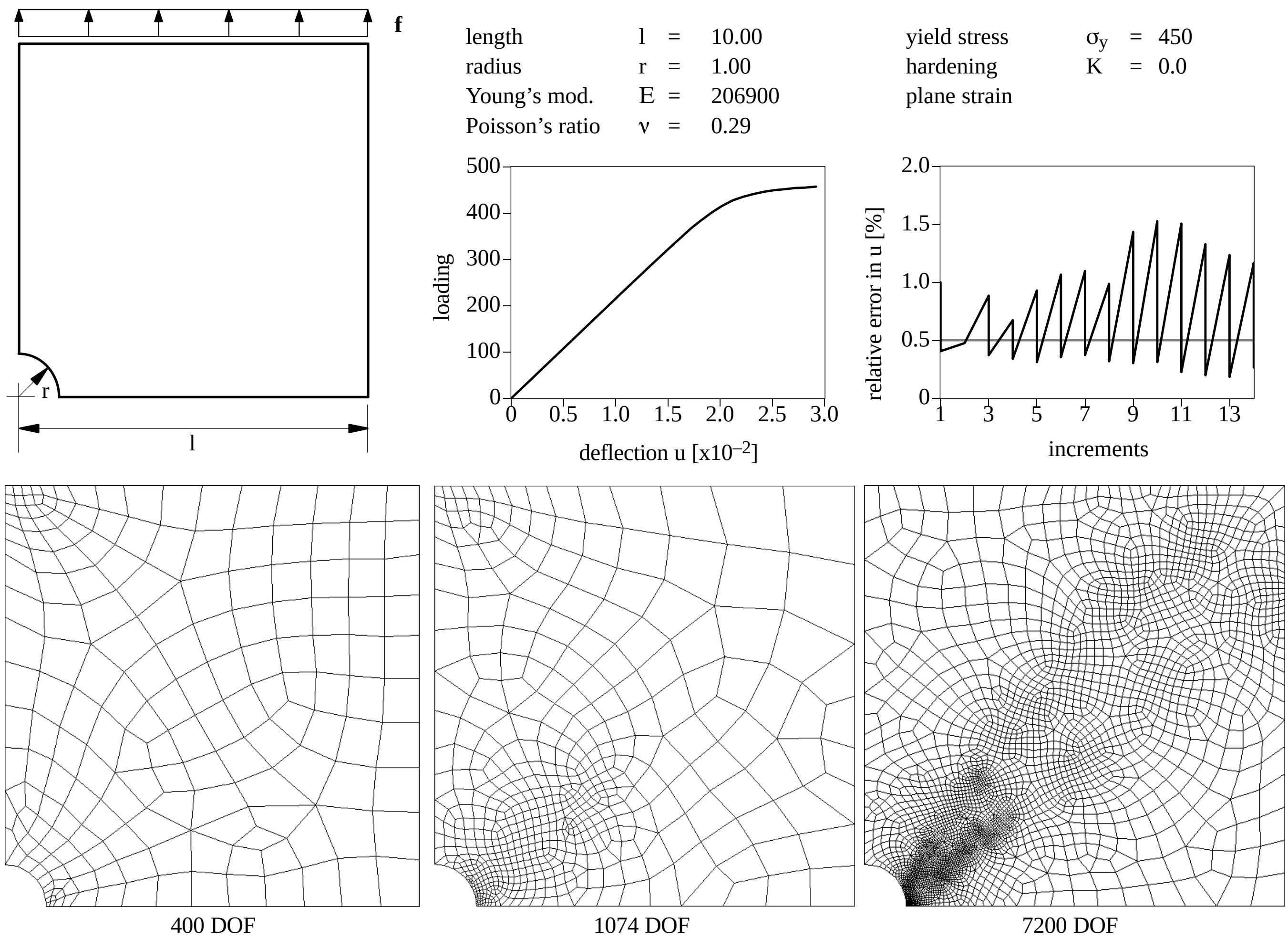

Figure 3: Perforated tension strip: Error control for vertical displacement

Acknowledgements

This work is part of the German Research Foundation (DFG) research project Ra 218/11-1 "Algorithms, Adaptive Methods, Elastoplasticity". The support is gratefully acknowledged.

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