A−posteriori error estimation and adaptivity for linear elasticity using the reciprocal theorem

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Abstract

The reciprocal theorem of Betti and Rayleigh or in other notions duality arguments are used to derive error estimators for the finite element approximation of various quantities, including local variables like single displacements and stresses. The proposed error estimator is evaluated solving the set of equations for an additional right hand side, simply applying the energy norm error estimators two times. Furthermore a general h−adaptive algorithm is introduced which allows to optimize meshes with respect to different user specified variables. The efficiency of the current approach is demonstrated for plate and shell examples.

KEY WORDS error estimation, adaptivity

1. Introduction

The significance of a−posteriori error control and adaptive algorithms for general finite element computations are meanwhile well known. For linear elastic problems several efficient error estimators have so far been developed. The most frequently used estimators are based on either residuals as initially proposed by Babuska and Rheinboldt [1] or postprocessed stresses as introduced by Zienkiewicz and Zhu [18]. These conventional a−posteriori error estimators and the related adaptive algorithms are mainly based on energy norm estimates. However for practical applications like stress analysis the globally defined energy norm is of limited importance. The relative errors in the variables of interest such as boundary reactions or point values of displacements differ in general from the global energy norm errors. The local errors can be estimated using duality arguments as recently introduced by different authors [3],[4],[13],[17]. The applied techniques are directly related to the influence line or surface concept of classical structural analysis and have in so far a very illustrative interpretation.

In the following the energy norm estimators are shortly recalled and an error estimator to control different globally or locally defined variables including single quantities is presented in detail. The proposed estimator is easy to evaluate and very efficient especially when a direct solver is applied in the finite element code. The original and the additionally defined dual problem are approximated by the same discretization and the energy norm errors of both problems are combined to a local error estimator. Numerical tests examine the accuracy and the efficiency of the present approach.

2. Governing equations

We briefly introduce the equilibrium equations in strong and weak form and define our notation. The equilibrium in strong form and the boundary conditions are given by

$$\begin{aligned} \operatorname{div}\, \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{p} &= \mathbf{0} \quad \text{on } \Omega \\ \boldsymbol{\sigma}(\mathbf{u})\tilde{\mathbf{n}} = \hat{\mathbf{f}} \quad \text{on } \Gamma_N \qquad \mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma_D \end{aligned} \tag{2.1}$$

where $\boldsymbol{\sigma}$ are the stresses, $\mathbf{u}$ are the displacements, $\mathbf{p}$ are body forces, $\hat{\mathbf{f}}$ are the tractions prescribed on the force boundary Γ_N, $\tilde{\mathbf{n}}$ is the unit normal vector to Γ_N and Γ_D is the displacement boundary. The weak form or the principle of virtual work can be obtained through multiplication of the equilibrium equations with a test function $\mathbf{v}$ and a subsequent integration by parts.

$$\int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v})\mathrm{d}x = \int_\Omega \mathbf{p}\mathbf{v}\mathrm{d}x + \int_{\Gamma_N} \hat{\mathbf{f}}\mathbf{v}\mathrm{d}x \tag{2.2}$$

Using appropriate finite element shape functions, the finite element approximation $\mathbf{u}^h$ to the displacements $\mathbf{u}$ can be computed in the standard way. For brevity, in the following we will use formal operator notation for eq. (2.2).

$$\mathrm{B}(\mathbf{u}, \mathbf{v}) = (\mathbf{p}, \mathbf{v}) + (\hat{\mathbf{f}}, \mathbf{v})_{\Gamma_N} \tag{2.3}$$

For shells the bilinear operator B(,) includes the virtual work of the membrane forces, bending moments and shear forces.

3. Brief review of energy norm error estimators

The approximation of the continuous displacements and test functions in the virtual work equation (2.2) by the finite element shape functions leads to discretization errors. The energy norm is a natural choice to measure the errors $\mathbf{e} = \mathbf{u} - \mathbf{u}^h$ due to its significance in the related extremum principles.

$$\| \mathbf{e} \|_e^2 = \mathrm{B}(\mathbf{e}, \mathbf{e}) = \int_\Omega (\boldsymbol{\sigma} - \boldsymbol{\sigma}^h) : (\boldsymbol{\epsilon} - \boldsymbol{\epsilon}^h)\mathrm{d}x \tag{3.1}$$

The errors in the energy norm can be estimated without referring to duality arguments as will be discussed in Section 4. Integration by parts of equation (3.1) yields the external work consisting of element internal residuals and jump terms across the element boundaries.

$$\int_{\Omega} (\boldsymbol{\sigma}-\boldsymbol{\sigma}^h) : (\boldsymbol{\epsilon}-\boldsymbol{\epsilon}^h) dx = \sum_{K=1}^{NEL} \left[\int_{\Omega_K} (\mathrm{div}\boldsymbol{\sigma}^h + \mathbf{p})\mathbf{e} \; dx + \int_{\Gamma_K \subseteq \Gamma_N} (\hat{\mathbf{f}}-\boldsymbol{\sigma}^h\tilde{\mathbf{n}})\mathbf{e} \; dx - \int_{\Gamma_K \not\subseteq \Gamma} \boldsymbol{\sigma}^h\tilde{\mathbf{n}}\mathbf{e} \; dx \right] \tag{3.2}$$

Here, NEL is the total number of elements, Ω_K is the domain and Γ_K are the edges of a finite element K. The first integral represents the element internal residuals of the finite element solution.

$$\mathrm{div}\; \boldsymbol{\sigma}^h + \mathbf{p} = \mathbf{R} \text{ on } \Omega_K \tag{3.3}$$

The second term represents the jumps in the tractions at the force boundaries.

$$\boldsymbol{\sigma}^h\tilde{\mathbf{n}}-\hat{\mathbf{f}} = \mathbf{J} \text{ on } \Gamma_K \subseteq \Gamma_N \tag{3.4}$$

For edges apart from the structure boundaries the jumps in the tractions of the both neighboring elements are the edge residual. The jumps are splitted by the factor 1/2 into the two neighboring elements K and K* sharing the considered edge.

$$\frac{1}{2}\left((\boldsymbol{\sigma}^h\tilde{\mathbf{n}})|_{K^*} + (\boldsymbol{\sigma}^h\tilde{\mathbf{n}})|_{K}\right) = \mathbf{J} \text{ on } \Gamma_K \not\subseteq \Gamma \tag{3.5}$$

The discretization errors can be computed applying the residuals **R** and **J** as loading to the structure as indicated by equation (3.2). The new problem has of course the same complexity as the original problem. For the estimation of the energy norm error it is common to solve this equation as a sequence of local problems including only a patch of one or a few elements loaded by the residuals. The boundary of the element patches can be fixed [1],[6] or loaded by equilibrated stresses computed from the finite element solution [8]. Note the mutual interaction of the local patches is not considered and leads to so called pollution errors. Non local effects can be significant for problems with singularities and heterogeneous material behavior and should be controlled as proposed by Babuska et. al. [3] or Oden et.al. [9].

An energy norm estimator can be derived following the recipes given by Johnson et. al. [6] The starting point is the error representation equation (3.2) in terms of the residuals.

$$\| \mathbf{e} \|_e^2 = \sum_{K=1}^{NEL} \left\{ (\mathbf{R}, \mathbf{e})_{\Omega_K} + (\mathbf{J}, \mathbf{e})_{\Gamma_K} \right\} \tag{3.6}$$

The interpolant $\Pi\mathbf{e}$ can be introduced into the error equation using the inherent orthogonality of the residuals to the finite element solution space, also denoted Galerkin orthogonality.

$$\| \mathbf{e} \|_e^2 = \sum_{K=1}^{NEL} \left\{ (\mathbf{R}, \mathbf{e}-\Pi\mathbf{e})_{\Omega_K} + (\mathbf{J}, \mathbf{e}-\Pi\mathbf{e})_{\Gamma_K} \right\} \tag{3.7}$$

Utilizing the Cauchy−Schwarz inequality the work of the residuals with the interpolation errors can be estimated by the L_2−norms of the residuals and the interpolation errors.

$$\| \mathbf{e} \|_e^{\,2} \leq \sum_{K=1}^{NEL} \left\{ \| \mathbf{R} \|_{\Omega_K} \| \mathbf{e} - \Pi \mathbf{e} \|_{\Omega_K} + \| \mathbf{J} \|_{\Gamma_K} \| \mathbf{e} - \Pi \mathbf{e} \|_{\Gamma_K} \right\} \tag{3.8}$$

The classical interpolation theory leads to the following bounds for the interpolation errors.

$$\| \mathbf{e} - \Pi \mathbf{e} \|_{\Omega_K} \leq C_1 h_K \| \mathbf{e} \|_{e,\Omega_{\tilde{K}}} \qquad \| \mathbf{e} - \Pi \mathbf{e} \|_{\Gamma_K} \leq C_2 \sqrt{h_K} \| \mathbf{e} \|_{e,\Omega_{\tilde{K}}} \tag{3.9}$$

where C_1 and C_2 are constants depending on the material behavior, the polynomial order of the shape functions and the element shapes. After inserting the interpolation estimates into the error equation (3.8) the energy norm error can be estimated by the residuals

$$\| \mathbf{e} \|_e^2 \leq \sum_{K=1}^{NEL} \left\{ \frac{h_K^2}{24Ep} \| \mathbf{R} \|_{\Omega_K}^2 + \frac{h_K}{24Ep} \| \mathbf{J} \|_{\Gamma_K}^2 \right\} \tag{3.10}$$

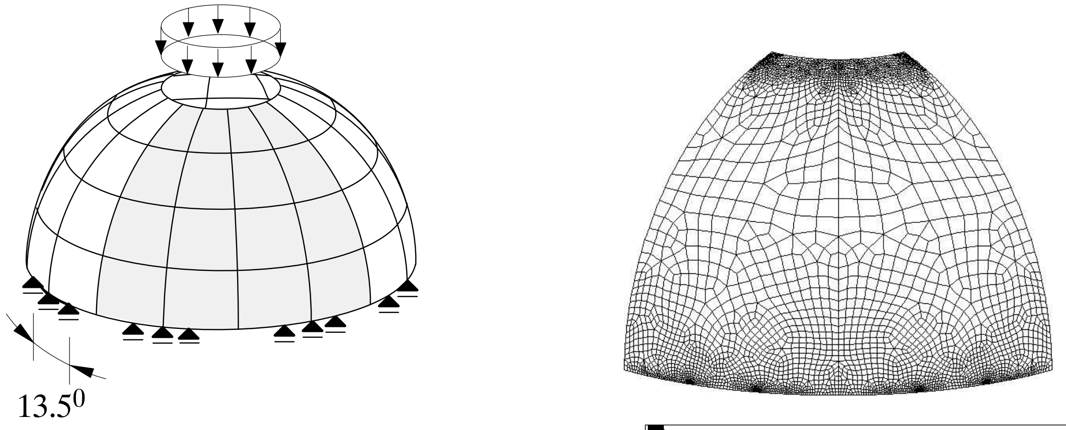
The interpolation constant C is chosen according to Kelly et.al. [7]. E is the Young's modulus and p the polynomial order of the shape functions. For shell problems the residuals consist of membrane, bending and shear parts. Instead of the elasticity constant E the respective membrane, bending and shear stiffness must be introduced in the interpolation estimates [12].

A different type of error estimators was proposed by Zienkiewicz and Zhu [18]. It is well known, that stresses at different points on a finite element mesh show different rates of convergence. With this information we can construct a stress field $\boldsymbol{\sigma}^*$ over the whole domain which has a higher accuracy than the finite element stresses $\boldsymbol{\sigma}^h$. The simplest method to determine an improved stress field is nodal averaging. It can be slightly improved by L_2 − projection of the discontinuous finite element stress field to the continuous finite element space. Both described methods show a limited performance especially for higher order elements. To remedy this Zienkiewicz and Zhu [19] proposed a patch wise least squares fit of a polynomial to the finite element stress field at discrete sampling points: the superconvergent patch recovery. All elements connected to the same node are combined to a patch. Across the patch a polynomial, which should have the same order as the shape functions of the displacement field, e.g. for biquadratic elements

$$\sigma_{ij}^* = a_0 + a_1 x + a_2 y + a_3 xy + a_4 y^2 + a_5 x^2 \tag{3.11}$$

is fitted in a least squares sense to the discrete sampling point stresses σ_{ij}^h. These points are the 2x2 Gauss points for biquadratic quadrilaterals.

$$\sum_{k=1}^{n} \left(\sigma_{ij}^* (\mathbf{x}_k, \mathbf{y}_k) - \sigma_{ij}^h (\mathbf{x}_k, \mathbf{y}_k) \right)^2 \rightarrow \min \tag{3.12}$$



radius	R =	10.0
thickness	t =	0.04
Young's mod.	E =	$6.825 \cdot 10^7$
Poisson's ratio	ν =	0.3

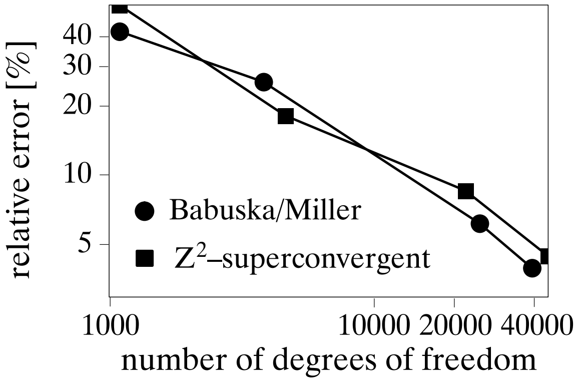


Figure 1: Partially supported hemisphere under vertical loading

For curved shells we apply the least square fit on the tangential plane of the current node. After computing the parameters of the polynomial (a_0, a_1, ..., a_5), the new Gauss point stresses can be calculated in a straight forward manner. The energy norm error can be computed by integration of the differences between the smoothed and unsmoothed stresses.

3.1. *Example*

As often mentioned the residual and the superconvergent patch recovery based estimators show similar performance in general. Both estimators are compared for a partially supported hemisphere discretized with nine node displacement elements (Figure 1). The lower edge is supported only in the vertical direction and the upper one is a free edge loaded by vertical loads. The relative error computed with both estimators shows the same behavior for this example.

4. Estimation of local errors

The initially proposed energy norm estimators are only appropriate for error control in a global sense. However for practical applications, like sizing of structural members, the variables of interest are mean or maximum values of stresses or displacements at some particular sections. The errors in local variables can be estimated using duality arguments as used in the past for a−priori analysis ("Aubin−Nitsche trick" [14]) or for computation of stress intensity factors [2]. In connection with adaptivity they have been introduced in a general framework by Johnson and co−workers [6],[5]. Lately the procedure was extended for error estimation and mesh refinement for local quantities (Babuska et.al. [3], Becker et.al. [4]). It is remarkable that the same approach was already described by Tottenham [16] at the beginning of the seventies. In the following we introduce the error estimator and the related refinement indicator for displacements and subsequently for stresses and integral variables.

4.1 *Error estimation for displacements*

As for energy norm error estimators, the starting point is the representation of the discretization errors depending on element internal residuals **R** and jump terms **J** (eqs. (3.3), (3.4) and (3.5)):

$$\begin{aligned} &\sum_{K=1}^{NEL}\left\{\text{div } \boldsymbol{\sigma}(\mathbf{e}) + \mathbf{R} = \mathbf{0} \text{ on } \Omega_K\right\} \\ &\sum_{K=1}^{NEL}\left\{\boldsymbol{\sigma}(\mathbf{e})\tilde{\mathbf{n}} = \mathbf{J} \text{ on } \Gamma_K \not\subseteq \Gamma_D, \quad \mathbf{e} = \mathbf{0} \text{ on } \Gamma_K \subseteq \Gamma_D\right\} \end{aligned} \tag{4.1}$$

In order to estimate the error of a specific displacement in the component i, e.g. at point $\mathbf{x} = \bar{\mathbf{x}}$, we additionally consider the following problem, also named the dual problem

$$\begin{aligned} &\text{div } \boldsymbol{\sigma}(\mathbf{G}) + \boldsymbol{\delta}_i(\bar{\mathbf{x}}) = \mathbf{0} \text{ on } \Omega \\ &\boldsymbol{\sigma}(\mathbf{G})\tilde{\mathbf{n}} = \mathbf{0} \text{ on } \Gamma_N \quad \mathbf{u} = \mathbf{0} \text{ on } \Gamma_D \end{aligned} \tag{4.2}$$

$\boldsymbol{\delta}_i$ is the load vector for the dual problem and its i−th component is a Dirac Delta (a single unit load) e.g. i=2

$$\boldsymbol{\delta}_2(\bar{\mathbf{x}}) = 0\cdot\mathbf{e}_1 + \delta(\bar{\mathbf{x}})\cdot\mathbf{e}_2 + 0\cdot\mathbf{e}_3 \tag{4.3}$$

where $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are the base vectors. The solution of the dual problem is the Green's function **G**. Applying the reciprocal theorem of Betti and Rayleigh to equation (4.1) and (4.2), we obtain the following relation for the error, which depend on the residuals **R** and **J** and the Green's function.

$$(\mathbf{e}, \boldsymbol{\delta}_i(\bar{\mathbf{x}})) = \sum_{K=1}^{NEL}\left\{(\mathbf{R}, \mathbf{G})_{\Omega_K} + (\mathbf{J}, \mathbf{G})_{\Gamma_K}\right\} \tag{4.4}$$

Note that the reciprocal theorem can also be derived by applying the divergence theorem two times

$$\begin{aligned} (\mathbf{e}, \boldsymbol{\delta}_i(\bar{\mathbf{x}})) &= -(\mathbf{e}, \text{div } \boldsymbol{\sigma}(\mathbf{G})) = (\nabla\mathbf{e}, \ \boldsymbol{\sigma}(\mathbf{G})) \\ &= (\mathbf{C}\nabla\mathbf{e}, \ \nabla\mathbf{G}) = \sum_{K=1}^{NEL}\left\{(\mathbf{R}, \mathbf{G})_{\Omega_K} + (\mathbf{J}, \mathbf{G})_{\Gamma_K}\right\} \end{aligned} \tag{4.5}$$

C is the material tensor. The term on the left hand side is the work of a unit load with the error function **e** and is equal to the error e_i of the i−th component of the displacement at point $\bar{\mathbf{x}}$.

$$(\mathbf{e}, \boldsymbol{\delta}_i(\bar{\mathbf{x}})) = e_i(\bar{\mathbf{x}}) \tag{4.6}$$

The present concept has an appealing engineering interpretation which may be illustrated by the following simple one dimensional example. A clamped bar with varying thickness is loaded partially by a uniform load (Figure 2). The exact displacements are logarithmic in the loaded part and constant in the unloaded part. The finite element approximation with two linear functions leads to errors that can be calculated in the following way. Inserting the finite element solution into the equilibrium equations the residuals are obtained. The discretization errors are the displacements and the stresses of the structure loaded by the residuals (Figure 3). For some reasons we are only interested in the tip displacements of the bar. The classical approach in structural analysis is to compute the influence line for this displacement and multiply it with the loads, i.e. in this case the residuals, which depend on the discretization (number of elements). The influence line **G** is computed by applying the dual variable to the tip displacement as a unit load to the structure (Figure 3).

$$e_i(\bar{\mathbf{x}}) = \sum_{K=1}^{NEL} \left\{ (\mathbf{R}, \mathbf{G})_{\Omega_K} + (\mathbf{J}, \mathbf{G})_{\Gamma_K} \right\}$$

Of course the solution of the dual problem is not known, but it can be computed also numerically based on the same discretization. The combination of the discretization errors for the initial and the dual problem leads to the error estimator of interest.

The formal approach is given in the following: The local error (4.4) can be expressed by the internal work

$$e_i(\bar{\mathbf{x}}) = \sum_{K=1}^{NEL} \left\{ (\mathbf{R}, \mathbf{G})_{\Omega_K} + (\mathbf{J}, \mathbf{G})_{\Gamma_K} \right\} = B(\mathbf{e}, \mathbf{G}) \tag{4.7}$$

Now the Galerkin orthogonality is used, with $\mathbf{G}^h$ as a finite element approximation of the Green's function.

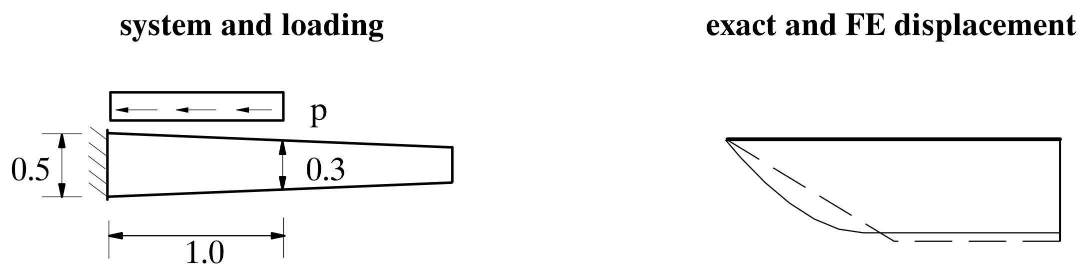


Figure 2: Clamped bar with varying thickness (geometry, load and solution)

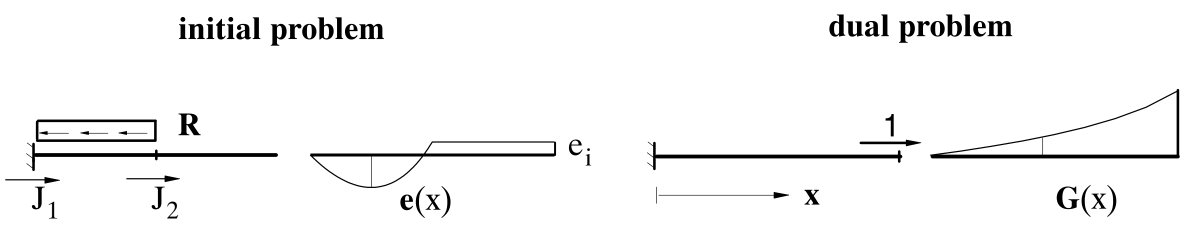


Figure 3: Definition of initial and dual problem (load and solution)

$$e_i(\bar{\mathbf{x}}) = B(\mathbf{e}, \mathbf{G}-\mathbf{G}^h) \tag{4.8}$$

Applying the Cauchy−Schwarz inequality

$$(e_i(\bar{\mathbf{x}}))^2 \leq B(\mathbf{e}, \mathbf{e})B(\mathbf{G}-\mathbf{G}^h, \mathbf{G}-\mathbf{G}^h) \tag{4.9}$$

the local error is estimated by the product of the errors in the energy norm of the initial or primal problem eq. (4.1) and the dual problem eq. (4.2). The inequality (4.9) can be computed elementwise

$$(e_i(\bar{\mathbf{x}}))^2 \leq \sum\nolimits_{K=1}^{NEL} B(\mathbf{e}, \mathbf{e})_{\Omega_K} \; B(\mathbf{G}-\mathbf{G}^h, \mathbf{G}-\mathbf{G}^h)_{\Omega_K} \tag{4.10}$$

This means that we can apply the residual or postprocessing based estimators introduced in the previous section to primal and dual problems and compute an estimate for the local error. This multiplicative procedure can be interpreted as follows: The second term representing the dual problem serves as weighting function ("ifluence line") and filters out the influence of the overall residuals over the displacement error of interest.

For second order differential equations and in two or three dimensions the internal energy of the structure loaded by point loads is infinite. More precisely the displacements are not in the Sobolev space H^1. The inequality is still true but has no practical importance. Therefore some sort of regularization must be applied in order to bound the energy. For this reason a distributed load $\mathbf{f}_i$ in a small region ω containing the point of interest $\bar{\mathbf{x}}$ is applied. The load vector is equal to one only in the i−th component and otherwise equal to zero e.g.

$$\mathbf{f}_2(\mathbf{x}) = 0 \cdot \mathbf{e}_1 + f_2 \cdot \mathbf{e}_2 + 0 \cdot \mathbf{e}_3 \quad \begin{cases} f_2 = 1 \text{ for } \mathbf{x} \in \omega \subset \Omega \\ f_2 = 0 \text{ for } \mathbf{x} \notin \omega \subset \Omega \end{cases} \tag{4.11}$$

The load $\boldsymbol{\delta}_i$ of the dual problem (4.2) is replaced now by $\mathbf{f}_i$ and Green's function is changed to $\mathbf{z}$.

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma}(\mathbf{z}) + \mathbf{f}_i(\bar{\mathbf{x}}) = \mathbf{0} \quad \text{on } \Omega \\ \boldsymbol{\sigma}(\mathbf{z})\tilde{\mathbf{n}} = \mathbf{0} \text{ on } \Gamma_N \qquad \mathbf{z} = \mathbf{0} \text{ on } \Gamma_D \end{aligned} \tag{4.12}$$

Applying again the reciprocal theorem to the primal and the dual problem yields

$$(\mathbf{e}, \mathbf{f}_i) = \sum\nolimits_{K=1}^{NEL} \left\{ (\mathbf{R}, \mathbf{z})_{\Omega_K} + (\mathbf{J}, \mathbf{z})_{\Gamma_K} \right\} \tag{4.13}$$

The first term represents the integral value of the error in the i−th component in the small region ω.

$$(\mathbf{e}, \mathbf{f}_i) = \int_\omega e_i dx \tag{4.14}$$

Notice that the mean value of the error in ω can be computed by dividing the integral through the area of ω. In practical computations the relative error η is only little effected by the regularization

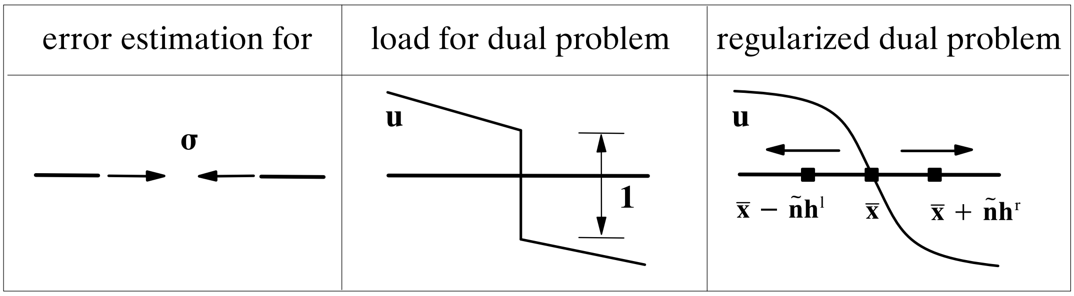


Figure 4: Loading of the dual problem

$$\eta = \frac{(\mathbf{e}, \mathbf{f}_i)}{(\mathbf{u}^h, \mathbf{f}_i)} \tag{4.15}$$

Due to the insignificant influence of the regularization $\mathbf{f}_i$ can be also chosen in a different way as defined above. In order to compute the error for the displacement of a specific finite element node, simply a nodal force can be applied. Utilizing the smearing effect of elements with a finite length and leading to a most simple way of regularization.

4.2 *Error estimation for stresses*

Utilizing the classical influence line/surface concept the errors in stresses or stress resultants can be estimated in a straight forward manner. In order to compute the errors for a stress variable at point $\bar{\mathbf{x}}$ a discontinuity on the related displacement variable must be applied in the dual problem problem

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma}(\mathbf{z}) + \frac{\partial}{\partial x_i}\boldsymbol{\delta}_j(\bar{\mathbf{x}}) &= \mathbf{0} \quad \text{on } \Omega \\ \boldsymbol{\sigma}(\mathbf{z})\tilde{\mathbf{n}} = \mathbf{0} \ \text{ on } \Gamma_N \qquad \mathbf{z} &= \mathbf{0} \ \text{ on } \Gamma_D \end{aligned} \tag{4.16}$$

If we apply now the reciprocal theorem and subsequently the divergence theorem, the error in the displacement gradient is

$$\left(\mathbf{e}, \frac{\partial}{\partial x_i}\boldsymbol{\delta}_j(\bar{\mathbf{x}})\right) = -\frac{\partial e_j}{\partial x_i} = B(\mathbf{e}, \mathbf{z}-\mathbf{z}^h) \tag{4.17}$$

Afterwards the errors in strains and stresses can be computed. Again in two or three dimensions it is not possible to apply a discontinuity to an individual displacement. Therefore a regularization must be applied. A simple strategy in the finite element method is to apply two loads with opposite directions at two neighboring nodes to point $\bar{\mathbf{x}}$ (Figure 4)

$$\operatorname{div} \boldsymbol{\sigma}(z) = \boldsymbol{\delta}_j(\bar{\mathbf{x}} + \tilde{n}_j h^r) - \boldsymbol{\delta}_j(\bar{\mathbf{x}} - \tilde{n}_j h^l) \tag{4.18}$$

h^l and h^r are the respective element lengths. The dual problems for stress resultants (e.g. moments) can be computed by similar regularizations.

4.3 *Error estimation for integral variables*

The above described concept for local variables can now be extended to arbitrary integral or global quantities. In the following some error estimators for selected integral quantities are introduced. The idea can be easily extended by mechanical reasoning to other important mechanical integral variables. Boundary tractions are often needed with a high accuracy for sizing. The dual problem for a specific boundary force component is defined by a movement of the boundary in the respective direction. In order to use the same finite element model for the initial and the dual problem the boundary movement can be applied as a regularized (smeared) displacement discontinuity in a small neighborhood of the boundary (Figure 5). Instead the smeared discontinuity can be imitated by a line load in the vicinity of the boundary. The errors in the boundary reaction V can be estimated in accordance to the previous sections by the energy norm errors for the initial and the dual problem.

$$|V|^2 \le \sum_{K=1}^{NEL} B(\mathbf{e},\mathbf{e})_{\Omega_K} B(\mathbf{z}-\mathbf{z}^h,\mathbf{z}-\mathbf{z}^h)_{\Omega_K} \tag{4.19}$$

Johnson and co−workers [6] give an error estimator based on duality arguments also for the L_2 −norm errors. The related dual problem is given by

$$\begin{gathered} \operatorname{div}\, \boldsymbol{\sigma}(\mathbf{z}) + \mathbf{e} = \mathbf{0} \quad \text{on } \Omega \\ \boldsymbol{\sigma}(\mathbf{z})\tilde{\mathbf{n}} = \mathbf{0} \ \text{ on } \Gamma_N \qquad \mathbf{z} = \mathbf{0} \ \text{ on } \Gamma_D \end{gathered} \tag{4.20}$$

The displacement error introduced as loading term on the right hand side can be computed by two discretizations with different accuracy[5]. Estimating the L_2 −norm errors, the importance of the dual problem is obvious. For the one dimensional example in Figure 2 the L_2 −norm is related to the length of the bar and to the residuals. Changing only the length of the unloaded region the residuals are not affected but the L_2 −norm is changed. Without considering the dual problem this difference cannot be represented. This also demonstrates, that the L_2 −norm errors cannot be estimated by local analyses based on residuals or superconvergence.

Furthermore the energy norm errors can be estimated without using the reciprocal theorem. Since primal and dual problem for the estimation of the energy norm errors are identical the reciprocal theorem has not to be applied, see discussion by Johnson and co−workers [6].

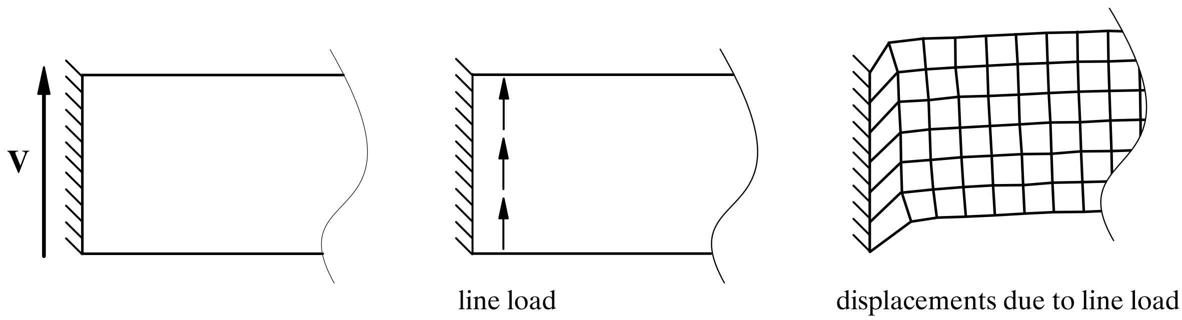


Figure 5:Regularized dual problem for the vertical boundary force V

5. Adaptive mesh refinement

Based on our mechanical interpretation, we can further conclude that the elementwise computed error contributions are also a good refinement indicator ρ_K.

$$\rho_K = \left(B(\mathbf{e},\mathbf{e})_{\Omega_K}\ B(\mathbf{z}-\mathbf{z}^h,\mathbf{z}-\mathbf{z}^h)_{\Omega_K}\right)^{\frac{1}{2}} \tag{5.1}$$

The regions with a big influence on the current displacement or stress, as indicated by the dual problem, should be more refined. This is also an essential feature of the proposed approach. In contrast to the global strong stability concept given by Johnson et.al. [6] the dual problem is also used for computing the influence of the individual elements on the specific variable.

The adaptive algorithm is based on the equidistribution of the element contributions ρ_K to the total error. This procedure gives the best results for minimizing the total energy norm error as shown by numerical experiments [10]. We assume the optimality also for mesh refinement with respect to other variables. The total permissible error is computed with a user specified relative error $\bar{\eta}$

$$\bar{\eta}(\mathbf{u}^h, \mathbf{f}_i) \tag{5.2}$$

$\mathbf{f}_i$ is the loading of the dual problem. Due to equidistribution the error value in each element should be

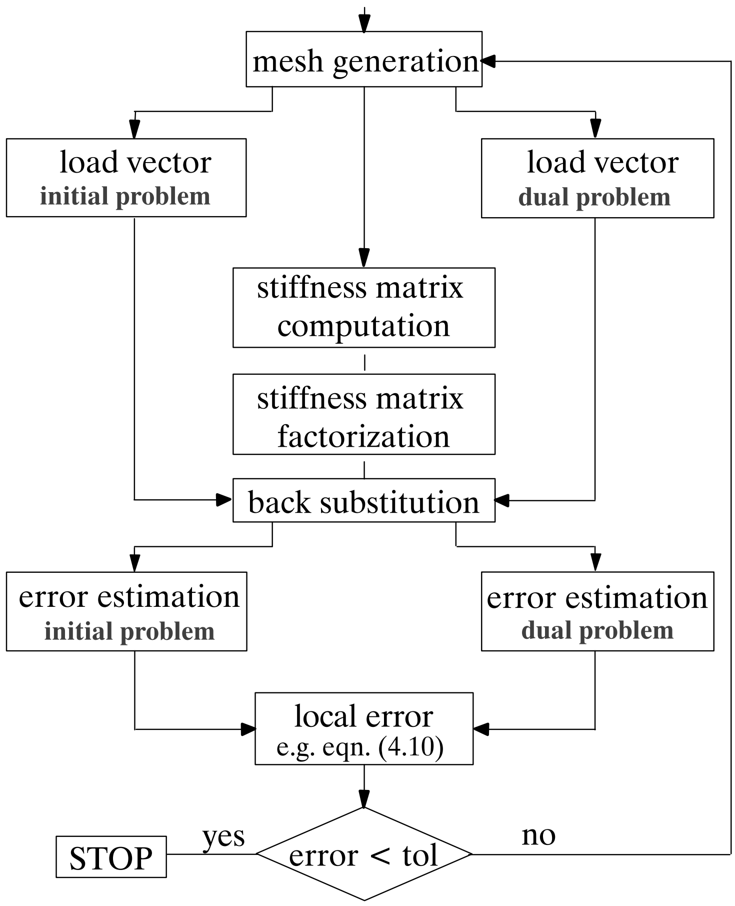


Figure 6: Adaptive algorithm

$$\bar{e}_m = \frac{\bar{\eta}(\mathbf{u}^h, \mathbf{f}_i)}{m} \tag{5.3}$$

where m is the number of elements. Further the following ratio is defined by

$$\xi_K = \frac{\rho_K}{\bar{e}_m} \tag{5.4}$$

For mesh refinement by element subdivision, the elements with large ξ_K values are refined. The number of refined elements in each step is limited, e.g. 20% to 40% of all elements. A more general method is to compute optimal lengths based on the a−priori convergence estimates for the errors. For smooth solutions the following estimate for the energy norm holds by uniform refinement of the elements

$$\| \mathbf{e} \|_e \leq Ch^p \tag{5.5}$$

whereby p is the convergence order and is equal to the polynomial degree p of the elements. The convergence order for the L_2 −norm of the displacements is equal to p+1. C is a constant independent of h and p. It is justified to assume the same convergence rates also for the element contributions ρ_K. Considering the displacement errors, the new element sizes can be computed by

$$h_{K,new} = \xi_K^{-\frac{1}{p+1}} h_{K,old} \tag{5.6}$$

For controlling stress errors the lower convergence rate should be taken into account

$$h_{K,new} = \xi_K^{-\frac{1}{p}} h_{K,old} \tag{5.7}$$

Based on the computed element lengths a new mesh is generated, for example applying the advancing front method. The whole algorithm as shown in Figure 6 is similar to the algorithms for error control by energy norms. Only load vector computation, equation solving and error estimation is done two times, namely for the initial and the dual problem. Applying a direct solver in the finite element code the stiffness matrix must be of course factorized only once.

6. Examples

The present approach is very general and can be used in different areas of structural mechanics to increase the effectivity and reliability of the finite element solution. We show two examples which demonstrate the capacity of the proposed algorithm.

A thin walled hyperbolic shell is loaded by vertical loads on the upper free edge. Part of the lower edge is supported in the vertical direction. The geometry and material data are given in Figure 7. Due to symmetry only one quarter of the shell is discretized with quadrilateral nine node shell elements. The error estimation and the mesh optimization is carried out for the vertical reaction force at the center of the boundary (point P). For error estimation of the initial and the dual problem superconvergent patch recovery,

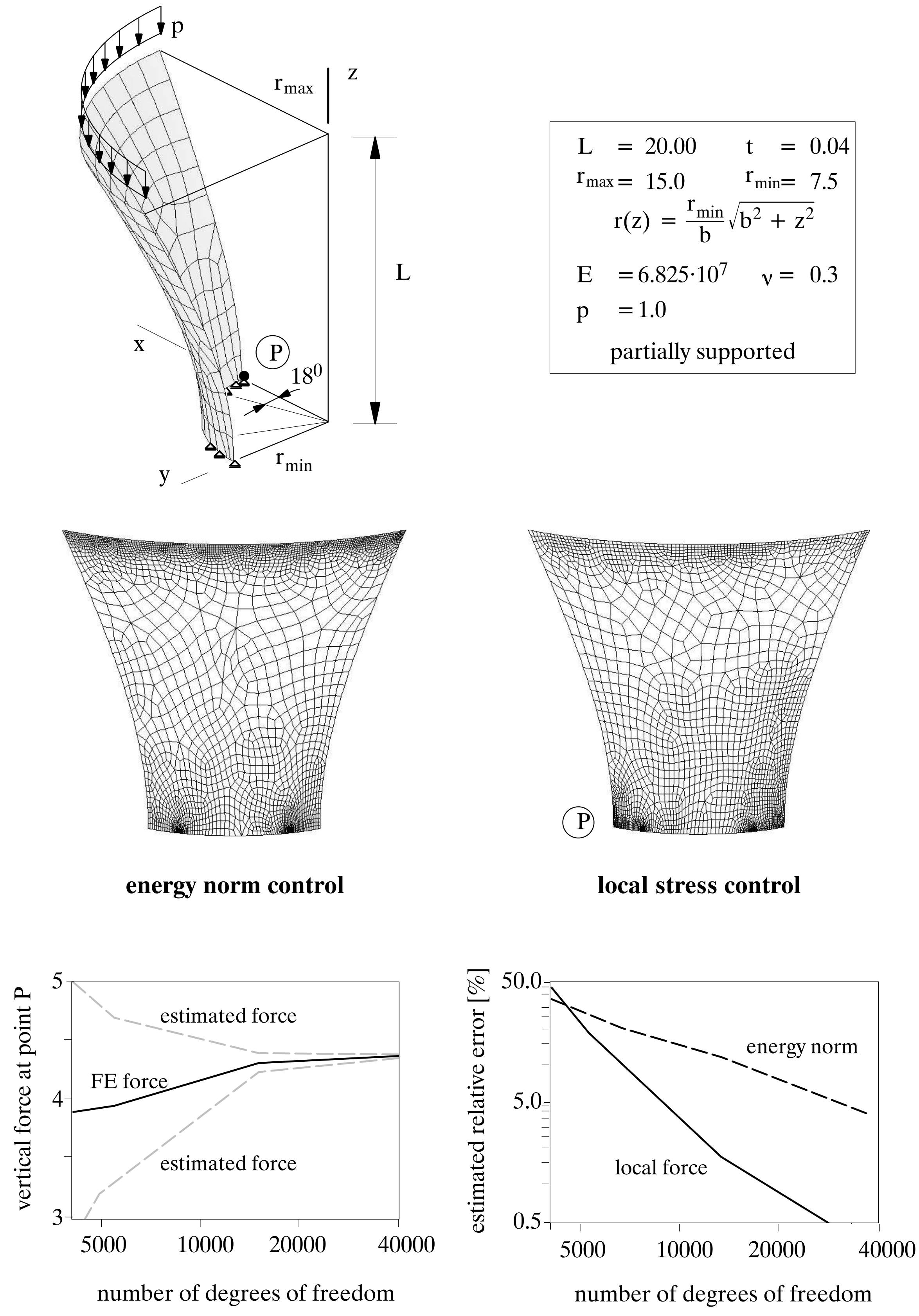


Figure 7: Axially loaded hyperbolic shell

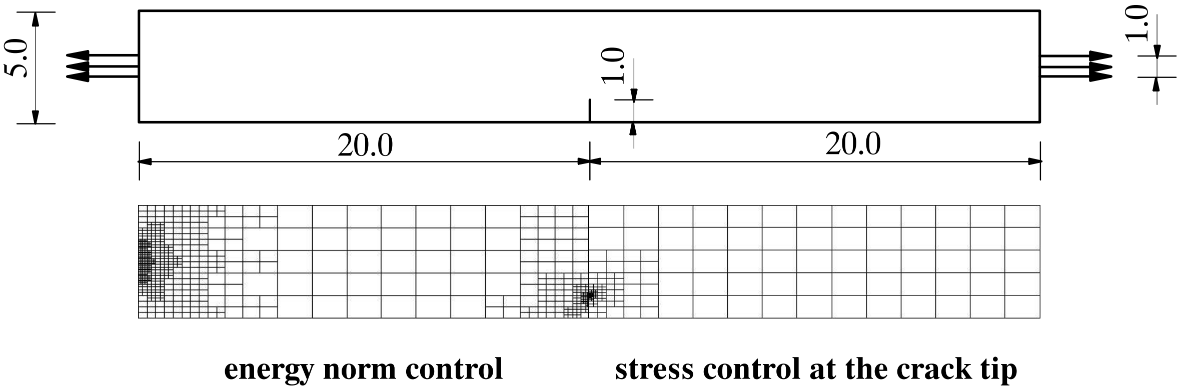


Figure 8: Cracked plate loaded by two concentrated forces

smoothing the stress field, is used. The loading for the dual problem consists of a line load applied in vertical direction in the vicinity of point P. The length of the line load is chosen as 1.0. As shown in the diagram (Figure 7) the estimated force gives an upper bound for the exact stress and is in good agreement with the converged finite element stress. Due to equation (4.9) the sign of the error of the force is not specified. The meshes optimized with respect to the vertical force and energy norm are quite different. Furthermore the error estimator indicates a different convergence rate for the boundary reaction force and the energy norm. For the specific boundary reaction force, the energy norm always leads to overrefinement.

The second example motivates the applicability of the present method to linear fracture mechanics. A cracked plate is loaded by two concentrated loads (Figure 8). Different methods for computing the stress intensity factors exist, some methods rely on the stresses close to the crack tip. We apply the following formula for the stress intensity factor K_I based on the analytical solution near the crack tip.

$$K_I = \lim_{r \to 0} \sqrt{2\pi r}\, \sigma_{xx} \tag{5.8}$$

σ_{xx} are the horizontal stresses in the distance r from the crack tip. Adapting the mesh for the horizontal stress at the node next to the crack tip renders the stress intensity factor 2.66 compared to the analytical solution 2.42 for an infinite plate. Local stress based adaptive algorithm refines the mesh correctly in the vicinity of the crack tip. It is more efficient than the energy norm based estimator, which also refines in other regions close to concentrated loads.

7. Conclusions

In this paper the relationship of the error estimators based on duality arguments to the influence line/surface concept frequently applied in structural mechanics is explained. Several a−posteriori error estimators are derived utilizing the well developed influence line/surface concept. Importantly, the dual problem is used for appropriate weighting of the separate contributions to the total error in contrast to the frequently used error estimators with only one global strong stability constant. Furthermore the element contributions are applied as a refinement indicator in connection with an h−adaptive algorithm.

An extension of the present methods to nonlinear problems in elastoplasticity is described by the authors [11] and by Suttmeier [15]. Here the linearized dual problem is used to compute the influence of the nonlinear residuals on the local errors.

Acknowledgment

This work is part of the DFG research project Ra 218/11–1 ”Algorithms, Adaptive methods, Elastoplasticity”. The support is gratefully acknowledged.

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A−posteriori error estimation and adaptivity for linear elasticity using the reciprocal theorem

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published in
Computational Methods in Applied Mechanics and Engineering
156, 352–362, 1998